

**PRINCIPLES OF PRODUCTIVE DISCIPLINARY ENGAGEMENT: FRAMEWORK
FOR DESIGN**

by

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This investigation explored the ways in which the four *principles of productive disciplinary engagement* (Engle & Conant, 2002) may be used as a tool for informing the design of the norms, structures, and classroom features that combine to form a learning environment that supports the CCSS-M. The study examined both the instructional practices employed by the teacher and the nature of student engagement in a suburban, regular education, seventh grade classroom over the course of one unit of study, following the implementation of intentional pedagogical practices aimed at implementing the four *principles of productive disciplinary engagement*. Data were gathered using several sources: transcriptions of video recordings of one unit of study that unfolded over 15 class sessions, the mathematical tasks used within the unit, lesson plans and teacher reflections, and a student survey. Applying an inductive scoring method (Miles & Huberman, 1994), the entire body of transcriptions of classroom video was scored in an effort to identify indications of each principle and the relationship between them. Teacher questions were scored using the Boaler & Brodie (2002) framework in an effort to identify the actions of the teacher that contributed to the enactment of the *principles of productive disciplinary engagement*. Mathematical tasks used throughout the unit of study,

considered to be an important element in achieving the principle of problematizing, were coded using the Math Task Analysis Guide (Stein & Smith, 1998). A survey was administered to students at the conclusion of the unit in order to understand their perceptions regarding the classroom environment and to triangulate the data. Evidence illustrates that elements such as the mathematical task in which students engage, utilizing the teacher-as-partner stance (Tabak & Baumgartner, 2004), deliberately offering students choices, and positioning students as capable, independent, decision-makers were identified among the ways the teacher encouraged the students to participate using the principles of productive disciplinary engagement. Results point to the interrelated nature of the four principles and student behaviors that occur when the social configurations are arranged so that students assume some of the roles typically associated with the teacher.

Keywords: productive disciplinary engagement, mathematical tasks, five practices for orchestrating discussion, participation pattern, teacher questions, noticing, teacher-as-partner

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PREFACE

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1.0 THE RESEARCH PROBLEM

1.1 INTRODUCTION

The National Council of Teachers of Mathematics (NCTM) has led an effort to change mathematics teaching and learning for more than twenty-years. Among the sources fueling the need for change was recognition that in an increasingly technological society, mathematics plays a central role and students were not being adequately prepared for this change. This lack of preparation is reflected in the analysis of results of the Third International Mathematics and Science Study (TIMSS) which compared Mathematics and Science achievement in fourth, eighth, and twelfth grade students from 41 nations. Only seven nations scored lower than students in the United States (Stigler & Hiebert, 1999). In addition, this study revealed that most U.S. students spend the majority of their instructional time completing procedural exercises. Our global society demands now that students are able to think, reason, and problem solve in addition to developing skills related to computational accuracy (Schoenfeld, 2013). Students are expected to understand mathematics not only as they master facts and procedures, but to see connections among multiple representations while building interpretive frameworks to make sense of their experiences (Engle, 2011).

The Common Core State Mathematics Standards (CCSS-M, 2010) provide an opportunity to reenergize the efforts of NCTM. Although these standards do not dictate

curriculum nor pedagogy, the emphasis they place on student reasoning and communication challenges the traditional method of delivery, wherein teachers model procedures and students use the procedures in repetitive fashion (Lampert, 1990; Ball, Goffney, & Bass, 2005). Supporting students in a way that encourages a belief in their own efficacy and a positive disposition toward mathematics, necessary for successful implementation of CCSS-M, demands teacher reflection regarding the vision of good instruction and the related classroom culture that supports it (Hill, Rowan, & Ball, 2005).

When one considers classroom culture, teaching mathematics in a way that is consistent with the Common Core State Standards includes more than teaching mathematical content. The Standards for Mathematical Practices are an integral part of the Common Core State Standards. The first three practices: make sense of problems and persevere in solving them; reason abstractly and quantitatively; and construct viable arguments and critique the reasoning of others, focus on making sense of problems and solutions through the process of logical explanation as well as through probing the understanding of others as students construct arguments, identify correspondences among approaches, and explore the truth of conjectures.

An environment that is supportive of these practices uses differences in student thinking as a tool for productive collective work (Boaler & Staples, 2008; Hufferd-Ackles, Fuson, & Sherin, 2004). Through the discursive patterns in the classroom, rights and obligations among participants are established, including expectations regarding the work of each participant as a member of the group. This talk, or classroom discourse, establishes a culture where students can participate or are marginalized in the learning and doing of mathematics. Established norms, such as the need to question peers' explanations, or provide mathematical justifications,

challenge the conventional assumptions about what it means to learn mathematics, as well as the expected role of the teacher and student.

It is through talk that mathematical ideas are aired, revised, connected to prior knowledge and to one another, examined, and challenged. Exchanges between students or teachers and students, go beyond describing a summary of steps in solving routine problems. Instead different strategies for solving challenging tasks are presented, differences in how problems are solved are expected and respected, and disagreements are resolved by reasoned arguments. Mathematical reasoning is seen as a practice to be learned, not an innate ability (Ball, Goffney, & Bass, 2005). Hence, patterns of participation in classrooms define learning.

This investigation explores the ways in which the four *principles of productive disciplinary engagement* (Engle & Conant, 2002) may be used as a tool for informing the design of the norms, structures, and classroom features that combine to form a learning environment. The study will examine both the instructional practices employed by the teacher and the nature of student engagement in a seventh grade classroom over the course of one unit of study, following the implementation of intentional pedagogical practices aimed at implementing the four *principles of productive disciplinary engagement* during the initial half of the year.

1.1.1 Learning as a social process

The type of environment which will support the implementation of the CCSS-M is based on the view that mathematics is learned not by the transmission of knowledge, but rather by participating in a culture as part of a “social practice” (Lave & Wenger, 1991, p. 47). Lave & Wenger emphasize the importance of social practices as defined by the culture, and view these practices as instrumental in learning mathematics. They stipulate that learning is always

“situated” in a community or culture. That is, the learner is naturally engaged because of the learning situation. Learning is seen as a complex, dynamic, social experience that occurs as situations of co-participation wherein the social context provides the context for the development of conceptual structures. From this perspective, when a novice engages in a new activity or in an unfamiliar body of knowledge, he becomes increasingly engaged and active as a member of the community. As he becomes increasingly knowledgeable he moves from the periphery as an apprentice, to eventually becoming a full participant in the practice that includes learning to talk, act, and interact in the manner of the community. Lave and Wenger (1991) refer to the process of moving from the periphery to a central position, *legitimate peripheral participation* (p. 29). Experts, who serve as models, guide the development of less expert members of the community. As the novice’s involvement increases, so does his mastery.

Rogoff (1994, p.209) contrasts learning as *transformation of participation* with discovery of knowledge by oneself, such as in child-run learning, and acquisition of knowledge from someone else, such as adult-run learning. She argues that what is learned is different; not that one model is better than another. Rogoff explains that the learner has a different relation with the information in each model. In the transmission model, “students learn information to be able to demonstrate that it has been encoded and retained in response to tests evaluating the transmission, piece by piece” (p. 210). In contrast, she notes that in collaboration with other children, “students learn the information, with purposes connected explicitly with the history and practices of the community” (p. 211). She emphasizes that both adults and children are actively engaged; adults are structuring shared endeavors while children learn to participate in the management of their own learning. Greeno (1997) views participation in social and cultural practices as that which defines learning mathematics, and the environment as that which defines

their adaptations related to participation. Lave & Wenger, Greeno, and Rogoff's views all emphasize the importance of the community related to student learning and the active nature of learning as a "two-sided" endeavor. Their theories point to the integral nature of social interaction and participation in the learning of mathematics.

Mathematical learning entails both social and communicative activities (Sfard, Forman, & Kiernan, 2001; Cobb, 1988). To consider teaching and learning it is imperative to bear in mind the culture of the learning environment. One of the constructs that link culture and learning is the notion of a *community of practice* (Lave, 1991; Lave & Wenger, 1991; Wenger, 1998). In Wenger's view, a community of practice characterizes the relationship among learners and stresses that individual learning takes place as each contributes to the norms and practices of the community. He emphasizes three essential dimensions that separate a *community of practice* from other, more common groups in which people participate. The first characteristic of practice that supports the coherence of its members is mutual engagement. Among the important ideas related to developing mutual engagement is the idea that people are engaged in "dense relations" of mutual engagement organized around something that matters to the group (Wenger, 1998, p. 74). Mutual engagement in this definition includes developing relationships through interactions that include both the latest piece of work-related information as well as the latest personal information. Group members are closely connected by more than just the task at hand. Mutual engagement then supports the negotiation of a joint enterprise, the second characteristic of practice that serves as a source of a community of practice. Disagreements are viewed as a productive part of the group's work. The enterprise is joint not because everybody believes the same thing, but rather because beliefs are communally negotiated. Negotiation is also central to the third characteristic of practice that includes accumulating a repertoire of

resources over time. These resources might include vocabulary, routines, gestures, or concepts that the community has adopted over time. Through the implementation of these three elements, according to Lave and Wenger (1991), a community of practice may be formed.

Communities of practice are not static. Sustaining the community involves developing mutual relationships, establishing who is good at what, who is easy and who is hard to get along with, aligning engagement with it, reconciling what the enterprise is about, producing or adopting tools, and inventing new terms, to name a few. Learning is seen to occur as the learners contribute to the evolution of communal norms and practices. In other words, when a learner vocalizes an argument he is simultaneously participating in a communal practice and an individual act. In the view of Sfard (2008) thought cannot be separated from communication and has coined the word *commognition* to reflect the intersection of the two.

By conforming to classroom norms, a student illustrates that she is a legitimate member of a community of practice that includes a particular social participation structure. Members of a community of practice negotiate their roles and hold each other accountable to work toward a common goal while using available resources. As students participate in conversations they take on certain roles, such as speaker, active or passive listener, or opponent of the issue at hand. As conversations evolve, roles and responsibilities change.

1.1.2 Creating supportive environments

If social relations and communication are considered to be essential elements of learning, then the environment that supports interaction must be carefully considered. Research has identified “design principles” or “principles of learning” that capture key theoretical ideas underlying innovative learning environments and provide guidance so that others can recreate them (Boaler

& Staples, 2008; Tarr et. al., 2008; Kazemi & Stipek, 2001; Chapin & O'Connor, 2007; Goos, 2004; Hiebert & Wearne, 1993; Silver, Smith & Nelson, 1995). Of particular interest to the study being proposed herein is the Productive Disciplinary Engagement framework (Engle & Conant, 2002). Presented as a theory, the principles of productive disciplinary engagement were proposed in response to a challenge to the design-based research community that included a request for a consensus on a small set of common principles that research suggested were critical for supporting effective learning environments. The principles of productive disciplinary engagement were presented as a proposal to members of the research community as a set of principles that they likely shared. Thus, the goal of Engle and Conant (2002) was “to abstract principles that could apply across learning environments in ways that could inform both the design of a wide range of new learning environments as well as research about existing ones” (Engle, 2011).

Consistent with the sociocultural and situative perspective of learning (Greeno, 1989, 1991; Lave & Wenger, 1991; Rogoff, 1994), Engle & Conant anchored their framework on the goal of “explaining students’ deep involvement in and progress on concepts and/or practices characteristic of the discipline they were learning about” (Engle & Conant, 2002, p. 400). As an organizer for thinking about instruction that supports productive disciplinary engagement, four principles were specified: authority, accountability, problematizing, and resources. These principles, described in the paragraphs that follow, provide the basis of the proposed study.

Authority reflects the idea that in order for students to become genuinely engaged in problems, they must have intellectual authority to do so. As learners are authorized to share their thinking, they become recognized as authors of the ideas and contributors to the ideas of others, leading to students becoming local authorities on a subject. In order to balance authority,

accountability addresses the need for students to be accountable to explain their own thinking; making sense of their own thoughts in light of other people's ideas. As accountability increases, learners improve their ideas so they are ready to be challenged more thoroughly by peers, internal authorities, and finally external disciplinary authorities (Engle, 2011). The assumption is that as a learner is expected to explain the reason that his ideas make sense given the relevant idea of others, the process provides the social conditions that prompt the learner to revise his ideas for the better. Other people's ideas become resources for revising, refining, and better defending one's own. Contrary to Engle and Conant's (2002) hypothesis, Forman and Ford (2014) hypothesize that the interpersonal process of constructing ideas through challenging peers precedes and fosters its intrapersonal appropriation. Notwithstanding this difference, both research groups agree on the importance of authority and accountability in the process of establishing productive disciplinary engagement.

A learning environment embodies the principle of problematizing to the extent that learners are encouraged to address problems that engender genuine uncertainty, are responsive to the learners' own commitments, and embody central aspects of the discipline. Problematizing can be achieved by creating uncertainty regarding what to do, what to conclude, or how to justify what one is doing. Providing resources, the fourth principle, provides balance to problematizing. The provision of relevant resources that are necessary for the work may be provided insufficiently, resulting in learners being overwhelmed with the problem at hand. In contrast, if too many resources are provided, the problematic nature of the work may be reduced so that the potential for productive disciplinary engagement is lost (Engle, 2011).

Research since the original work that introduced the principles of productive disciplinary engagement has been extensive (Windschitl & Thompson, 2006; Gresalfi, Hand, & Hodge, 2006;

National Research Council, 2008). Engle (2011) reviewed seventeen case studies that were explained using the principles of productive disciplinary engagement. The work to date suggests that the principles of productive disciplinary engagement appear to capture some consensus ideas within the research community related to a wide variety of respected educational innovations developed over the last twenty years (Forman, Engle, Venturini, & Ford, 2013). However, the work to date provides little guidance to teachers regarding ways to operationalize these ideas in the classroom. Articulating the knowledge and skills necessary for creating the kind of learning environment that implementation of the principles of productive disciplinary engagement demands has yet to be defined.

1.1.3 The purpose of the study

This investigation explored the ways in which the four principles of productive disciplinary engagement may be used as a tool for informing the design of the norms, structures, and classroom features that combine to form a learning environment. The study examined both the instructional practices employed by the teacher and the nature of student engagement in a seventh grade classroom over the course of one unit of study, following the implementation of intentional pedagogical practices aimed at implementing the four principles of productive disciplinary engagement during the initial half of the year. The guiding assumption is that for most students, the extent of their engagement in personal thought and the thinking of peers defines their learning. Further it is assumed that when all four of the principles of productive disciplinary engagement are realized together in the learning environment, productive disciplinary engagement has been achieved.

Learning to talk with peers regarding the discipline is critical, and depends on specific teacher practices to encourage this kind of behavior. Although Engle & Conant's work provided a synthesis of design features that were highlighted in individual research studies, this study adapts their framework as a practical tool for use by a classroom teacher in the design of the learning environment. Supporting teachers in a way that enables them to encourage student learning by creating environments that foster communication and mathematical reasoning, consistent with the CCSS-M, Mathematical Practices calls for a great deal of learning on the part of teachers. Transforming teachers' knowledge, beliefs, and habits of practice will require professional development that can lead to changes in the judgments and complex decisions that teachers make on a moment-by-moment basis. If opportunities to develop new levels of awareness and knowledge are to be provided, research that decomposes effective practices and positions them in a way that professional developers may present them to teachers will be crucial to the successful implementation of the consensus of ideas that research on this subject has captured (Grossman, Hammerness, & McDonald, 2009).

1.2 THE RESEARCH QUESTIONS

This study examined the instructional practices and the nature of student participation in a seventh grade mathematics classroom over the course of one instructional unit in the second half of a school year, following the implementation of intentional pedagogical practices aimed at implementing the principals of productive disciplinary engagement during the initial two quarters of the year.

The study examined the following research questions:

Research Question #1:

In what ways are the principles of productive disciplinary engagement: 1) evident in the instructional practices implemented by the teacher and 2) enacted by the students?

- A) In what ways does the teacher expand or constrict the distribution of authority within the classroom? In what ways do the students act with authority?
- B) In what ways does the teacher hold students accountable to themselves, peers, and the discipline? In what ways do the students engage in the social and intellectual practices that reflect accountability?
- C) In what ways does the teacher encourage students to take up intellectual problems that simultaneously: engender genuine uncertainty in students, and embody some central aspects of the discipline in question, that defines problematizing? In what ways do the students reflect genuine uncertainty in the instructional environment?
- D) In what ways does the teacher encourage students to amplify their capacity to solve problems through the provision of resources? In what ways do students utilize resources to problem solve?

Research Question #2

- A) What work is required of the teacher in order to translate the principles of productive disciplinary engagement into practice?
- B) What challenges and successes does the teacher encounter along the way?

1.3 SIGNIFICANCE

Key features of innovative instructional environments that have been captured by Engle & Conant's (2002) principles of productive disciplinary engagement offer a theoretical framework for designing supportive learning environments. The study will provide insight into the extent to which the use of the framework accomplishes this goal. Hence, results of the study may serve a practical purpose in guiding others interested in the design of learning environments. It extends the work of Engle & Conant (2002) by employing the framework, originally created as a tool for research, as a tool for the design of learning environments by practitioners. The study may be of particular interest at the present time, as teachers struggle to enact mathematics instruction consistent with the eight Mathematical Practices; an integral component of successful implementation of the CCSS-M.

1.4 LIMITATIONS

There are several limitations to the study. First, the study includes only one teacher/researcher and her students in a suburban setting. Because of the small number of participants enrolled in the study, generalizability of its findings is limited.

In addition, the teacher is a doctoral candidate in Mathematics education who is simultaneously serving the role of researcher. Her undergraduate education in engineering and her work in that field prior to becoming an educator, influences her perspective of both mathematics and mathematics education. Her background is as atypical of mathematics teachers as are her strong beliefs regarding ways that students learn (Stodolsky & Grossman, 1995).

Further, the teacher/researcher was free to design lessons using any appropriate curricular resources rather than being confined to one particular publisher, a restraint often imposed on teachers. These differences further limit the generalizability of the data.

1.5 OVERVIEW

This document consists of five chapters. The first chapter provided an argument about the essential nature of social relations and communication in learning; demanding that the environment that supports interaction be carefully considered. Further, it proposed the potential use of the principles of productive disciplinary engagement as a tool for informing the design of the norms, structures, and classroom features that combine to form a learning environment that supports that talk. Chapter 2 provides a detailed account of the research that has contributed to the development of “design principles” or “principles of learning” that capture key theoretical ideas underlying innovative learning environments. Because the principles of productive disciplinary engagement were proposed as a consensus of ideas shared by the research community related to the design of learning environments, the work described in Chapter 2 tests the efficacy of using these principles in this way. Chapter 3 presents the methodology and includes a description of the context of the planned study, the participants in the study, the data sources, and the analysis procedures. The results of the analysis are reported in Chapter 4. Chapter 5 presents the discussion of the findings, conclusions drawn from these findings, and suggestions for future research.

2.0 THE LITERATURE REVIEW

2.1 INTRODUCTION

The purpose of this study is to consider the use of Engle & Conant's (2002) principles for productive disciplinary engagement as a practical tool for the design of a learning environment. If one is to repurpose these principles, it is critical to recall that they were proposed in response to a challenge to the research community that included a request for a consensus on a small set of common principles that research suggested were critical for supporting effective learning environments. The principles of productive disciplinary engagement were presented as a proposal to members of the design-based research community as a set of principles that they likely shared. Thus, the goal of Engle and Conant (2002) was “to abstract principles that could apply across learning environments in ways that could inform both the design of a wide range of new learning environments as well as research about existing ones” (Engle, 2011).

Consistent with the sociocultural and situative perspective of learning (Greeno, 1989, 1991; Lave & Wenger, 1991; Rogoff, 1994), Engle & Conant anchored their framework on the goal of “explaining students’ deep involvement in and progress on concepts and/or practices characteristic of the discipline they were learning about” (Engle & Conant, 2002, p. 403). They considered social relations and communication to be essential elements of learning, and the environment that supported that interaction worthy of careful consideration. As an organizer for

thinking about the environment that supports productive disciplinary engagement, four principles were specified: authority, accountability, problematizing, and resources. Research has indicated that all four elements must be embodied in the environment to realize productive disciplinary engagement, and that having one or more principles missing results in productive disciplinary engagement falling short (Engle, 2004; Engle & Faux, 2004 as reported in Engle, Conant, & Greeno, 2007).

2.2 METHODS

The literature review includes empirical studies related to the teaching and learning of mathematics and science from 1980 onward. Key search terms included forms of mathematical thinking, mathematical discourse, small collaborative mathematics groups, mathematical learning, principles of productive disciplinary engagement, classroom discourse, collaborative mathematical problem solving, and small group interaction. I limited my review to include those studies that address K-12 classrooms, not focused on the use of technology. Although I have read literature outside the field of mathematics that informs my perspective, I have included only studies directly related to mathematics and science teaching and learning. I limited the search to research published in scholarly journals or as book chapters in the field of education. I also performed a search within key journals that publish research in Mathematics Education, such as the *Journal for Research in Mathematics Education*. Examining the references for the related research as well as conducting Google Scholar searches on selected citations led to other relevant articles for inclusion. Studies that were not written in English were excluded.

2.3 ORGANIZATION OF THE LITERATURE REVIEW

Research studies in two related areas have direct relevance to the proposed study. In this chapter I describe studies that provide a valid argument for using the *principles of productive disciplinary engagement* as a lens for examining features of the environment that contribute to student engagement. The review of these studies assumes that Engle and Conant (2002) were correct in their assumption that the four *principles of productive disciplinary engagement* are necessary for productive disciplinary engagement, and examines selected research for the ways that these principles are apparent in the learning environment under study.

In order to answer that question, I examine four studies that focus on whole class settings to provide the reader with insight regarding the strength of the evidence that underlies these principles as key elements in the creation of the learning environments described. The review serves to analyze selected studies in terms of the ways in which the principles of productive disciplinary engagement were enacted in the intervention as well as to review the outcomes and limitations of each study. Next, I examine research that focus on the ways that the students and teachers participate in the classroom learning environment.

First I define the key terms that are prevalent in the research. Then I review the literature that informs the design of the proposed study. The literature review is accomplished in three sections: research related to whole group discussion, research related and the ways that people participate in the instructional environment, and a summary of research directly related to the *principles of productive disciplinary engagement* since the original research was reported.

2.4 PRINCIPALS OF PRODUCTIVE DISCIPLINARY ENGAGEMENT

2.4.1 Definition of key terms

Fostering norms that support a learning environment conducive of productive work by all its members has been explained using the concept of “productive disciplinary engagement” (Engle & Conant, 2002), which the authors define as, “students’ deep involvement in and progress on concepts and/or practices characteristic of the discipline they were learning about” (p. 400). The authors define engagement using three criteria. First, the number of students participating is indicative of engagement. That is, more students participating, and few students “off task” is considered as more engagement. Second, greater intensity in the way students participate in the mathematics instruction is greater engagement. Such intensity might be apparent as students’ speech overlap and the way they attend to each other with eye gaze and body position. Third, the extent to which participation of learners is responsive to others indicates greater engagement. Examples of responsive behaviors might include students making emotional displays, building on the thinking of others, and attending to their work for long periods of time. Further, Engle & Conant (2002) define engagement to be disciplinary when there is “some contact between what students are doing and the issues and practices of a discipline’s discourse” (p. 402). They define the word productive to include, “significant disciplinary progress from the beginning to the end” of students’ engagement (Engle, 2007, p. 215). They believe that productivity largely depends on the discipline, the task, the topic, and where students are when they begin to address the problem. Productivity, then, can only be judged on an individual basis. Disciplinary progress could be related to a design, making a new connection between ideas, or students shifting from explaining their own ideas toward a posture that allows them to compare and challenge others’

ideas. These principles are intended to function as organizers for thinking about norms established by the teacher in the learning environment.

A skeletal definition of each of the constructs that comprise the principles of productive disciplinary engagement as well as examples of their existence in research will be provided in the following paragraphs. Engle and Conant (2002) define authority with regard to two ideas. The first idea is related to students having an agency in defining addressing and resolving problems. The second includes members of the learning community positioning students as stakeholders by publicly identifying them with the claims, approaches, explanations, designs and other responses to problems. Students may develop into classroom experts to whom others rely for help. In other words, students who have authority are encouraged to be authors and producers of knowledge rather than consumers of it. In other words, students become active learners who take responsibility for their own learning (Hufferd-Ackles, Fuson, & Sherin, 2004). It demands that teachers share authority with students in developing the learning community, and in so doing provide the opportunity for students to develop a sense of agency.

This idea is congruent with the goals of the NCTM's Principles and Standards document (1989, 2000), and the CCSS-M Mathematical Practices (CCSS-M, 2010). Central to these standards is the commitment to develop mathematical literacy and power in every student wherein mathematical power encompasses the ability to "explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems" and the self-confidence and disposition to do so (National Council of Teachers of Mathematics, 1989, p. 5). Similarly the Common Core State Standards', Mathematical Practices emphasize the importance of active learning in supporting all students (CCSS-M, 2010). The first three practices: make sense of problems and persevere in solving

them; reason abstractly and quantitatively; and construct viable arguments and critique the reasoning of others, focus on making sense of problems and solutions through the process of logical explanation as well through probing the understanding of others. Students must construct arguments, identify correspondences among approaches, and explore the truth of conjectures. These practices demand that students have the authority to actively engage in the learning process. Through these principles, the Mathematical Practices encourage students to be active knowers and doers of mathematics. In terms of Engle & Conant's (2002) principles, the principle of authority is congruent with these NCTM Principles and CCSS-M Mathematical Practices.

Research points to evidence of teacher moves that encourage students to be accountable to the teacher and other members of the learning community, through the implementation of classroom norms (Yackel & Cobb, 1996). This principle, being accountable to others and to disciplinary norms, implies that the teacher and other members of the learning community foster students' responsibilities to consult others in constructing understanding in a domain; it doesn't require acceptance of others' views, but responsiveness to them. "This principle is an expression of the value that each member of a learning community is not an authority unto himself, but one intellectual stakeholder among many in the classroom and beyond" (Engle & Conant, 2002, p. 405). Students who take their peers' ideas into account may be better positioned to persuade others of their own ideas, thus motivating further participation. In addition, being held to disciplinary norms helps to balance student authority and reduce the chance of students constructing haphazard responses to problems without peer review (Cobb & Hodge, 2002).

Engle & Conant (2002) discuss the importance of "problematizing" as the third core idea in their framework. Engle (2011) describes problematizing as, "any individual or collective action that encourages disciplinary uncertainties to be taken up by students" (p. 6). She further

describes problematizing to include the extent to which genuine uncertainty is engendered in students, that problems are not easily resolved, that problems embody “big ideas of the discipline”, and that they are related to a topic that is of some interest to the learner. In order to succeed in problematizing, a teacher must create an environment where students must persevere together toward a common goal. Discourse among students is truly necessary in an environment that embodies the principle of problematizing because a course toward solution is not apparent. Students genuinely need to talk in order to determine a solution path, draw a conclusion, or synthesize their work. Problematizing describes a purposeful choice by the teacher in terms of the kinds of tasks students will engage and how problems will be designed. In other words, problematizing includes choosing tasks that encourage students to both interpret them and persevere in solving them, using available knowledge and resources. Genuine uncertainty must be created within students to have enacted the principle of problematizing. Congruent with other research that draws a connection between discursive participation, the related teacher practices that influence student learning, and the mathematical task selected by the teacher, problematizing is a central theme (Leinhardt & Steele, 2005; Stein, Smith, Henningsen, Silver, 2000; Silver, Smith, & Nelson, 1995; Smith, 2000; Lotan, 2003; Hiebert & Wearne, 1993; Kieran, Forman, & Sfard, 2003).

Research collectively points to the importance of the task in creating a sense of uncertainty in students. A mathematical task is defined as a set of problems or single complex problem that focuses students’ attention on a particular mathematical idea (Stein, Grover, & Henningsen, 1996). It is the task that provides something worthy of talk while promoting or discouraging students to explore deeply the intended mathematical goal. Although task selection

and problematizing are not synonymous, problematizing largely depends on task selection and the enactment of the task by the teacher.

Problematizing is balanced by the principle of providing resources to students. With insufficient resources, students are unable to act and may be overwhelmed with the challenge; with too many resources provided, the problematic nature of the task is diminished. Engle & Conant (2002) describe the provision of resources as a necessary fourth element in the support of productive disciplinary engagement. They define resources very generally and include anything or anyone that may be seen as necessary to support the embodiment of the other principles. Resources may be as fundamental as providing students with time to solve meaningful problems (Henningsen & Stein, 1997) or may be more specific to the task. They cite examples of providing resources; the provision of home-based modes (talk that is consistent with the style at home) of discussion in support of problematizing content as well as the provision of models and norms in the classroom. Peers, physical manipulatives, teacher questions, and anything that might amplify a student's capacity to problem solve would qualify as a resource using this broad definition.

Encouraging productive disciplinary engagement through the implementation of the four principles as I have described them is to characterize a new participant structure in terms of social practices and the related discursive practices (Palincsar & Brown, 1984; Tabak & Baumgartner, 2004) . When I speak of participant structure, I adopt the definition described by Cornelius & Herrenkohl (2004). These authors describe participant structures to include what some others call participant frameworks (O'Connor & Michael, 1996); that is an interest in conventional classroom social arrangements, including rights and responsibilities, with the notion of "social positioning" or the ways that discussions linguistically place speakers in

relation to the subject matter and other participants (Goffman, 1974). Although “positioning theory” as a means to explain the relationship between discourse and psychological phenomena is beyond the scope of this work, attending to the dynamic role that participants assume within classroom conversations is relevant to this study. Therefore, positioning provides a useful tool in describing and assessing relationships.

I have chosen eight studies to examine in detail with regard to assessing the extent to which the principles of productive disciplinary engagement are present in learning environments that mathematics research has deemed to be effective. Four of these studies represent the analysis of student learning in whole group settings, and four studies focus on small, collaborative groups. I chose the studies based on several factors: 1) the study includes the researchers’ attention to both the teacher and students actions, 2) the study uses a very “up close” examination of the culture which places the researcher in a position to provide an in-depth portrait of the individuals and their relationships in the learning environment, 3) the study illustrates a relationship between the instructional environment and the way it contributes to student success, and 4) the studies chosen represent different sample sizes from case studies to large-scale studies. The research that I have included in the first two sections of this chapter are summarized in Tables 2.1. Table 2.1 serves to illustrate a few examples of the presence of the principles of productive disciplinary engagement that are evident in studies that examine whole group interactions . The table is not intended to be a comprehensive list of every example of the principles productive disciplinary engagement present in the studies.

2.5 PRODUCTIVE DISCIPLINARY ENGAGEMENT IN WHOLE GROUP

DISCUSSION

In the following section, four samples that represent examples of research wherein the instructional environment has been described to be effective are reviewed with regard to the presence or absence of the principles of productive disciplinary engagement. In each study, I have attempted to both describe the study, as well as its relationship to each of the principles of productive disciplinary engagement.

2.5.1 Boaler & Staples (2008)

In a 5 year, longitudinal study of high school students employing mixed methods, Boaler & Staples (2008) studied student learning in approximately 300 students at Railside school and two other high schools having approximately the same size but composed of students of different demographics. While Railside was an urban high school with an ethnically, linguistically, and economically diverse student body, Hilltop was situated in a rural setting wherein half of the students were Latino and half white, and Greendale included a high majority of white students. The schools also represented differences in choice of curriculum. Both Hilltop and Greendale employed traditional curriculum while Railside used a reform-oriented approach. The authors use the word reform-oriented and reform teaching to imply the use of curricula and pedagogy that is consistent with the principles promoted by the National Council of Teachers of Mathematics (NCTM, 1989). The unique features of Railside school included a commitment to reform teaching that include mixed-ability groups, and opportunities for student advancement as opposed to a traditional tracking system. The comparison groups in the study were

approximately 300 students who followed the traditional curriculum and pedagogy at Hilltop and Greendale, and 300 students at Railside who were taught using reform-oriented approaches. The same content was taught in all three schools.

Data sources included: lesson observations, interviews, videos, questionnaires, and assessments that combined to provide information on the teaching and learning practices in the different approaches and the students' responses to them. The researchers assessed students' understanding of math content using content-aligned tests and open-ended project assessments that were reviewed by teachers prior to administering. Groups were videotaped as they worked on open-ended projects. Researchers also gathered scores on state assessments that they used to compare performance of the three target schools.

The authors report many features that combined to create an environment wherein differences in attainment between students of different ethnic groups were reduced or erased. I have tabulated some of the features of the instructional environment that provide evidence of the principles of productive disciplinary engagement in Table 1. Among those features that the authors attribute to student success were that teachers presented all students, in a heterogeneous setting, with a rigorous curriculum that included tasks of high cognitive demand and high expectations for success. Drawing conceptual connections using student's existing understanding, modeling high level performance, and pressing for justifications and explanations, were just a few of the teacher moves that this research has identified as essential elements in Railside students' success and which is supported by other research (Carpenter, Fennema, Peterson, Chiang, Loef, 1989; Carpenter & Fennema, 1992, Kazemi & Stipek, 2001). With regard to the principles of productive disciplinary engagement, choosing tasks of high cognitive demand contributes to problematizing; creating genuine uncertainty in students.

Pressing for justifications and explanations is a way to hold students accountable to rigorous thinking, while drawing conceptual connections and modeling high-level performance is an example of encouraging the use of resources.

Teachers at Railside taught students to be accountable not only for their own ideas, but for the each others' learning. At Railside students talked about the value that their peer-group added to their own learning, but comments were distinctly reciprocal, and expressed concern for the learning of classmates as well. One of the ways in which teachers nurtured the feeling of accountability was through the assessment system. Teachers assigned grades for individual and group tests and for the quality of conversations that groups had. Another way in which responsibility was encouraged was through the practice of asking one student a question after the group had engaged in a task. If the student couldn't answer the question, the teacher would leave the group to encourage further discussion among students and return later to ask the same student the question again. In the intervening time the group was expected to be helping the student learn the mathematics in question. In this way, teachers and peers attended to students' current understanding and worked together toward creating high quality work and deep conceptual understanding.

Pre and post tests were used to gauge the mathematical progress of students. At the beginning of year 1 the students at Railside were achieving at significantly lower levels than students at the two other schools using the traditional teaching method ($t=-9.141$, $p<0.001$, $n=658$). At the end of year 1, students were administered a test of algebra to measure what students had learned over the year. The difference in means (1.8) indicated that the students were performing equally. At the end of year 2, students were administered a test of Algebra and Geometry. Railside students significantly outperformed the students in the traditional

classrooms ($t=-8.304$, $p<0.001$, $n=512$). Boaler & Staples concluded that the combination of heterogeneous grouping, high-level tasks, and instructional moves of the teachers, which included purposeful teacher questioning and accountability, combined to create an equitable environment that supported increased student learning.

The Boaler & Staples (2008) work presents large-scale evidence of improvements in student learning resulting from instructional practices and the learning environment in the classroom. There is substantial evidence for these claims because the data sources are triangulated (Miles & Hubbard, 2004). Standardized test scores are supported by content paper/pencil tests and video footage of student interactions in the classroom. The authors seek to explain the classroom environment that supported student learning using a variety of elements and attempt to understand the interaction of many variables that combine to form that environment. The principles of productive disciplinary engagement are clearly present in the learning environment that supported student learning.

2.5.2 Chapin & O'Connor (2008)

The essential nature of student accountability was investigated by Resnick, O'Connor, Michaels, and Chapin (Chapin & O'Connor, 2004; Chapin & O'Connor, 2007; O'Connor & Michaels, unpub; O'Connor & Michaels, 1993; Michaels, O'Connor, & Resnick, 2008). Deemed Accountable Talk, several studies involving Language Arts and Mathematics instruction demonstrated the potential student learning gains when teachers use specific norms and forms of talk that actively engages learners in reasoning about the knowledge they are acquiring. The term Accountable Talk refers to several techniques used by teachers to encourage respectful and reasoned discussion in a classroom setting. The use of Accountable Talk moves encourages

student participation in this discussion through three Accountable Talk features: accountability to knowledge, accountability to the community, and accountability to rigorous thinking. Talk that is accountable to the community creates an environment that provides students with opportunities to formulate their own ideas and challenge the ideas of others. Participants listen carefully to one another's ideas and provide reasons for agreement or disagreement. Drawing of reasonable conclusions and making logical connections is the focus of talk that is accountable to standards of reasoning. "Talk that is accountable to knowledge, is based explicitly on facts, written texts or other publicly accessible information that all individuals can access" (Michaels, O'Connor, & Resnick, 2008, p. 289). The teacher's goal is to guide student discussions toward academically correct concepts and ideas.

Chapin & O'Connor (2004) conducted qualitative research on the ways effective teachers used various discourse practices and utterance types to orchestrate classroom discussion in elementary and middle school classrooms. Using the Connected Mathematics and Investigations curricula in a low-income, urban school district the research, called Project Challenge, began with the intent to identify and develop unrecognized talent and the potential for "giftedness in mathematics." Project Challenge teachers were supported in using a variety of academically productive "talk moves" designed to press students to explicate their reasoning and build on one another's thinking. Project Challenge students included 100 students each of four years, beginning in their 4th grade year. Students were chosen for the project using the Naglieri Non-Verbal Abilities Test (NNAT) among approximately 600 third grade students. Those with scores in the 7th, 8th and 9th stanines were automatically placed in the program. The remaining half of the 100 places were filled with students scoring in the 5th and 6th stanines, the average scorers. Instruction was provided five days per week and was centered on complex problems that

prepared students for college-track math courses in high school. After only eight months in the program, students were administered the MCAS, the Massachusetts Comprehensive Assessment System test. On average, 57% of each 4th grade Project Challenge class, scored "Advanced" or "Proficient" on the MCAS mathematics test; significantly better than Massachusetts scored as a whole (38%). Project Challenge results on the MCAS in sixth grade were even more dramatic. At the end of 6th grade, for students who had participated in the project for three years, at least 82 % of each Project Challenge class scored "Advanced" or "Proficient" on the MCAS mathematics test. The California Achievement Test Mathematics Portion (both Concepts and Computation) was administered to students when they reached sixth grade. The results were scored externally. Each year, the cohort being tested scored on average at the 90th percentile on this nationally-normed test.

In order to compare the performance of students in the Project Challenge and those who participated in traditional instruction, a post hoc, quasi-controlled comparison of students who had been eligible for Project Challenge (and matched with Project Challenge students), but not selected, was conducted. The differences between MCAS scores of the Project Challenge students and their matched controls was significant and effect sizes were large (Cohen's $d = 1.8$) (O'Connor & Michaels, 2007; Resnick, 2007).

This study represents a substantial body of research that reflects the idea that learning is robust when learners are held accountable for becoming actively involved in reasoning (Resnick & Hall, 1998). The principle of problematizing was embodied through the use of carefully selected tasks, that provided something of substance in which students could engage. The study encouraged the use of resources through the encouragement of talk between students; they used each other as ideational and relational resources. Further, students were free to author their own

ideas and to build on the thinking of others, leading to the enactment of the principle of authority.

2.5.3 Goos (2004)

A similar set of priorities is echoed in a case study by Goos (2004) wherein she considered what specific teacher actions might contribute to a culture of inquiry in a secondary mathematics classroom. She examined a single classroom over a period of two years in an effort to analyze some of the teaching and learning practices used by one teacher in helping students appropriate the ways of knowing, speaking, and acting that is characteristic of a community of mathematical inquiry. This study is indicative of a case study wherein the researcher explores a bounded system over time using a detailed, in depth collection of data (Cresswell, 2007).

The participants in the study were white, middle-class Australian students in grades 11 and 12. Weekly lesson observations were supplemented by video and audio- taped recordings of teacher-student and student-student interactions and field notes made during lesson observations. Stimulated recall interviews were conducted with the teacher and groups of students to seek their interpretations of videotaped excerpts. Semi-structured interviews of students were also conducted to investigate their views about learning mathematics. The author's effort to triangulate the data brings credence to her claims.

Goos reports several features of the classroom that are significant examples of providing students with authority. First, discussions in this classroom were frequently directed between students, rather than through the teacher, in contrast to a traditional classroom. The teacher allowed students to seek and receive help from each other as he purposefully ceded control of debates. Further, the teacher regularly allowed time in class for students to study worked

examples so that they would learn to find their way independently through mathematical texts and be able to use them as resources. Frequently, during disagreements, the teacher withdrew from discussion and encouraged students to resolve issues alone; a move to encourage accountability and relinquish authority. His classroom was described as one where students engaged in tasks for extended periods of time, consistent with the principle of problematizing, and as they engaged their knowledge claims were recognized as conjectures that had to be validated. Explaining was used to both evaluate and strengthen student understanding. Students asked questions of peers; proposing and evaluating alternate solutions to mathematical tasks.

The rich description of the classroom interactions in the Goos (2004) study makes a valuable contribution to the mathematics education literature because it offers a view into a way that the culture of a mathematical community might be developed over time. It highlights several assumptions, teacher actions, and resultant student actions that contribute to the development of the community. Goos (2004) moves from the assumption that there is a way to establish a community with specific features that are desirable in supporting student learning, and then goes about to find out how the community is established from the start. Relinquishing authority by the teacher is a significant feature in this study, although all of the principles of productive disciplinary engagement are apparent. The researcher has claimed that the instructional environment provided to students has had a profound impact on the classroom culture and the ways that students have engaged in learning mathematics. Although she does not have student test scores as a data source to support her claim, she effectively uses description of student behaviors to indicate positive examples of learning.

2.5.4 Silver, Smith & Nelson (1995)

One seminal project that supports the careful selection of tasks and in so doing provides an excellent example of the *principles of productive disciplinary engagement* was QUASAR (Silver, Smith & Nelson, 1995). QUASAR was both a practical demonstration project and complex research study catalyzed by the inequities in mathematics learning opportunities in schools serving poor communities. It was a large-scale study that focused attention on particular ways that teachers mediate curriculum approaches to make them equitable. For the purposes of this review, it provides a large-scale example of a research project wherein problematizing is fundamental to student success.

Using specific criteria, researchers chose six public, middle schools in different states as sites for the study. The students in these schools included a culturally and socially diverse, urban population. The project's goals were to, "develop, implement, and refine innovative instructional programs for all students" (Smith, 2000, p. 354). In support of these goals, the work of the researchers included curriculum development and modification, professional development and teacher support, classroom and school-based assessment design, as well as community outreach efforts over a period of five years.

In an analysis of roughly 150 tasks used over a three-year period in the QUASAR project, Stein, Grover, & Henningsen (1996) determined that over three fourths of the mathematical episodes included tasks intended to invoke students' reasoning, conceptual understanding, and problem solving. Tasks that were part of the intervention were designed to encourage students to use novel methods of problem solving and rather sophisticated mathematical thinking in QUASAR classrooms. Teachers had a more difficult time enacting the tasks at high level, with only 42% of cognitively challenging tasks enacted in ways that maintained students opportunity

for engagement in ways that demanded high-level cognitive processes. Another distinguishing feature of the instruction was the frequency of opportunity for students to communicate and collaborate. Many teachers emphasized the development of understanding through student discourse about mathematical ideas using both collaborative learning groups and whole class discussion.

The extent to which the instructional design had beneficial effects on student learning were evaluated based primarily by measuring changes in student mathematical performance over time using the QUASAR Cognitive Assessment Instrument (QCAI). Developed specifically to examine students' capacity to reason, problem solve, and communicate mathematically, the QCAI challenged students to solve complex mathematical tasks requiring the use of high-level thinking (Lane, 1993; Silver & Lane, 1993). Further, analysis that examined the relationship between instruction and learning in QUASAR classrooms indicated that student gains were especially positive in classrooms wherein the setup and implementation of instructional tasks encouraged high-level thinking and reasoning, the use of multiple representations, mathematical explanations, and multiple solution strategies. In addition, eighth grade students who participated in QUASAR were administered a subset of items from the NAEP Grade 8 mathematics assessment in an effort to establish normative information regarding QUASAR students. QUASAR students significantly outperformed the NAEP disadvantaged urban sample, the sample demographically most similar to QUASAR students (Silver & Lane, 1995).

Among the outcomes of the QUASAR project include a qualification system, called the Math Task Analysis Guide (Appendix C), that uses the cognitive demand necessary to solve the task to rank tasks. The framework is important because it provides a practical means that teachers might use to choose tasks worthy of classroom use. The cognitive demand, that is the

level and type of thinking that a task has the potential to engage in a student, has been differentiated into four distinct categories (Stein & Lane, 1996). The Math Task Analysis Guide illustrates characteristic features of tasks requiring low-level and high level cognitive demand. In general, low-level cognitive demand tasks are algorithmic in nature (Stein, Smith, Henningsen, & Silver, 2000). They involve using or producing previously learned facts or procedures. There is little ambiguity about the direction or steps needed for solution and they are generally not connected to concepts underlying the procedure. The focus is primarily on obtaining a correct answer with little need for explanation. Examples of tasks requiring low cognitive demand would include completing a two-digit by two-digit multiplication problem using an established procedure, or reproducing memorized addition facts. Conversely, tasks that require students to explore and understand mathematical concepts, processes, or relationships, requiring that students develop meaning through the use of multiple representations and analysis, while accessing prior relevant knowledge fall into the category of high-level cognitive demand. These tasks often require students to use non-algorithmic thinking while persevering to develop solution strategies.

In terms of Engle & Conant's principle of using resources, the use of multiple representations offers opportunities to engage in using ideational resources. While one representation may encourage sense-making in one student, another may amplify student problem solving capacity in another. Freedom to use multiple representations also embodies the principle of authority; offering the students the chance to author solutions themselves. As students authored solutions, teachers pressed for conceptual understanding; an instructional practice that is indicative of the principle of accountability. Choosing a task that created genuine uncertainty in students, a criteria necessary for the principle of problematizing, contributed to

student learning as well. All four of the principles of productive disciplinary engagement were clearly present in this study.

Using data from the QUASAR study, Stein & Lane (1996) examined the relationship between student learning and the way that tasks were enacted during instruction. Specifically, using video tapes and narrative summaries of classroom observations from four school sites, the work examines differences in instruction at sites that varied in levels of student learning gains (high, medium, or low). The researchers theorized that the sites identified as having high student learning gains would be characterized by instruction that included set up and implementation of the task using instructional features promoted by the NCTM Principles and Standards (1989). A stratified random sample of the 620 tasks used was chosen using year, site, and teacher as stratification dimensions. The resulting sample included 144 tasks. Each task was coded along four main categories including task description, task set up, task implementation, and the factors of maintenance or decline of high-level tasks. In order to examine the consistency with which tasks were set up and implemented, the characteristics of the tasks as they were setup were superimposed with the characteristics of the task as implemented, resulting in the ability to directly compare the two. Evidence of student learning outcomes was based on results of the assessment tool developed by QUASAR researchers, the QUASAR Cognitive Assessment Instrument, QCAI. School sites were rank-ordered based on the student learning gains over a three-year period.

The results indicate that classrooms that focused on tasks of high-level cognitive demand were associated with the most gain in student learning. Conversely, students' learning gains were relatively small where instruction was procedurally based. Further, students' learning gains were more robust when the instruction was focused on a task that was set up to include a high level of

cognitive demand, even if it was implemented in such a way that students were not engaged in high levels of reasoning or problem solving. Student thinking in the high set-up/high implementation classrooms was characterized by complex, non-algorithmic thinking while student thinking in the low set-up/low implementation classrooms was mechanical, and included predetermined routes that exposed neither the conceptual underpinnings nor mathematical reasoning. The different forms of thinking were associated with differences in student learning based on the QCAI.

These QUASAR-related studies point to the significance of the task as selected and enacted by the teacher as a critical step toward student learning. Tasks that require students to explore and understand mathematical concepts, processes, or relationships, requiring that students develop meaning through the use of multiple representations and analysis, while accessing prior relevant knowledge fall into the category of high-level cognitive demand. These tasks often require students to use non-algorithmic thinking while persevering to develop solution strategies. Properly chosen tasks that contribute to the principle of problematizing offer the opportunity for everyone to make a contribution to both individual and group success. Although important, the high level mathematical tasks themselves are necessary but not sufficient in developing a rich learning environment because the cognitive demand of the task may change as the tasks are enacted during instruction (Stigler & Hiebert, 2004; Tarr et al., 2008; Stein & Lane, 1996). Maintaining the task's high level of cognitive demand without allowing the task to degrade to a routine, algorithmic problem requires vigilance and skill on the part of the teacher. Drawing conceptual connections, modeling high level performance, pressing for justifications and explanations, are just a few of the elements that must be integrated into the

classroom by the teacher if the cognitive demand is to be maintained and the *problematizing* principle is to undergird instruction (Stein & Lane, 1996).

The four studies that are summarized in Table 1 represent a sample that illustrates the presence of the *principles of productive disciplinary engagement* in learning environments that mathematics research has deemed to be effective. Engle and Conant (2002) presented their *principles of productive disciplinary engagement* as a theory that these principles could apply across learning environments in ways that could inform both the design of a wide range of new learning environments as well as research about existing ones. Based on these examples of research that represent different-sized studies and with varying character, I concur that the *principles of productive disciplinary engagement* are present in innovative learning environment.

Table 1. Principles of Productive Disciplinary Engagement in Whole Class Discussions

Study	Authority	Accountability	Problematising	Resources
Boaler & Staples (2008)	Valuing student-developed solutions Student reporting of their own sense-making	Pressing for justifications and explanations Individual and group assessments Questioning. Returning to a group to check understanding.	Tasks of high cognitive demand Focus on conceptual understanding.	Drawing conceptual connections Teachers encouraged students to use peers as resources.
Chapin & O'Connor (2004)	Students were free to author ideas.	Talk moves aimed at pressing students to explicate their reasoning and build on one another's thinking.	Use of Connected Mathematics and Investigations curriculum. Focus on conceptual understanding.	Teacher moves encouraged students to engage in the thinking of peers.

Table 1 (continued)

Goos (2004)	Discussion was directed between students. The teacher frequently withdrew from discussions and encouraged students to resolve issues alone.	Knowledge claims were conjectures that had to be validated. Explaining was used to both evaluate and strengthen student understanding. Students asking questions of peers. Students proposing and evaluating alternate solutions to mathematical tasks.	Students attending to single tasks for extended periods of time.	The teacher regularly allowed time in class for students to study worked examples so they would find their way independently through mathematical texts and be able to use them independently. Teacher modeled mathematical thinking.
Silver, Smith & Nelson (1995), Stein & Lane (1996); Stein, Smith, Henningsen & Silver (2000)	Multiple solutions were encouraged.	Pressing for mathematical explanations and justifications.	Tasks of high cognitive demand.	Multiple representations were encouraged. Teacher modeling high level performance.

2.6 PARTICIPANT FRAMEWORKS

The principles of productive disciplinary engagement inform the design of a classroom culture that utilizes a particular participant structure. Contrary to an IRE sequence (Cazden, 1988; Mehan, 1979) in a traditional classroom, where the teacher initiates a question, waits for a student's response, then reserves the right to evaluate the student's response, the sharing of authority that is an essential principle of productive disciplinary engagement, has the potential to dramatically transform the roles and responsibilities of classroom participants. Researchers have proposed and evaluated new classroom "participant structures" (Phillips, 1972) that have enabled students to become contributors and active participants in classroom discourse. Many of these studies point to the complicated process of language use in teaching and learning and the ways that language may be used as a resource by a teacher to coordinate participant structures (Erickson, 1982; O'Connor & Michaels, 1993; Lampert, 1990; Boaler & Brodie, 2004; Hufferd, Ackles, Fuson, & Sherin, 2004; Cornelius & Herrenkohl, 2004; Tabak & Baumgartner, 2004).

Lampert's (1990) action research study designed to examine whether and how it might be possible to bring what it means to know mathematics in school, closer to that within the discipline by deliberately altering classroom discourse and tasks is an example of research that points to the ways in which the teacher's discursive practices work to coordinate academic tasks with social participation. In this study, Lampert served the dual role of researcher and teacher in a fifth grade, heterogeneous classroom in a public school. The study included an intervention that included purposefully changing the mathematical tasks and social norms that were present in a traditional classroom; challenging students' assumptions about what it means to know mathematics. She illustrates that students have changed in the way they think about what it means to know and do mathematics. She uses the term "cognitive technologies" to

mean the knowledge of tools, symbols, and vocabulary that were necessary in establishing the mathematical arguments that were a part of her goal for students. In her description of her teaching she relied heavily on “cognitive technologies” as students gained both knowledge of mathematics (mathematical content), and knowledge about mathematics. She described pedagogical moves involved in implementing new participant structures, including encouraging students to engage in struggling through solving of mathematical tasks. The questions that she expected them to answer went well beyond simply determining if they could get the answer. She expected them to answer questions about mathematical assumptions and the legitimacy of their strategies. This study is an early example wherein the goal was in essence to subject theory to the conditions of practice. Lampert clearly represents herself as having different skills than an elementary teacher. However, notwithstanding the differences, her claim that it was possible to produce the kind of classroom environment that was more consistent with the discipline, and advocated by the standards-based movement was warranted. The purposeful modification of a traditional IRE participant structure to one wherein authority was shared and roles and responsibilities were consequently redefined, produced a classroom culture that would be congruent with one informed by the principles of productive disciplinary engagement.

This study is widely cited because it provides insight into what is possible. Through her knowledge of theories and conviction toward developing a culture wherein mathematics learning was more than memorization and procedures, she demonstrated that what had been recently reported in the NCTM Principles and Standards (1989) could become a reality.

Likewise, Boaler & Brodie’s (2004) questioning framework delineates differences in the focus of questions from finding the right answer toward a posture of questioning to uncover the mathematical thinking behind the answers. Central to the framework is the concept that different

question types shape the nature and flow of classroom discourse and that instruction must be viewed using a fine grain size to uncover essential differences in instruction. Details of the Questioning framework may be found in Appendix B. Their work draws attention to role of the teacher question type in determining participant structure. For example, a question that is aimed at gathering information results in the teacher likely serving in the role of an evaluator of the information, whereas a question posed with the intent of encouraging discussion likely places the students in a more authoritative position, resulting in a different participant structure.

Tabak & Baumgartner (2004) use the metaphor of symmetry to compare participant structures. They present the teacher as partner versus the teacher as monitor and the teacher as mentor structure. These three participant structures were used to delineate the ways that teachers engaged students in conversations about their work in a small group setting. In a symmetrical, teacher as partner, relationship the teacher investigated with students, joining a group for a few minutes and taking part in the investigation as a genuine member of the group. In this way the teacher-student relationship was described as being symmetrical. This shift in the mode of interaction, from a traditional classroom includes a shift in number of teacher or student turns. There may be a sequence of consecutive teacher or student turns. This symmetrical relationship is in direct contrast to the teacher as monitor participant structure wherein the teacher serves to set up the task and make sure the flow of classroom activity is sustained. The teacher either briefly acknowledges student progress or she may provide feedback and explain procedures. In the teacher as mentor structure the discourse pattern is an I-R-F (Initiation, response, feedback from the teacher) pattern. The teacher tries to help the students align their thinking and actions without dictating actions or explanations. Interactions focus on supporting the substance of the inquiry process. In my opinion, the teacher as mentor structure is inadequately defined in relation

to the others. For this reason, I will refer to only the teacher as monitor or teacher as partner participation summary.

2.7 RESEARCH RELATED TO THE PRINCIPLES OF PRODUCTIVE DISCIPLINARY ENGAGEMENT

Following Engle & Conant's (2002) proposal of the principles of productive disciplinary engagement as a framework that was intended be used across classroom case studies to support comparisons, much research has built on the original framework. Engle (2011) advances the ideas introduced in the framework to a large extent through the review of fourteen case studies that were explained using the principles, three case studies that included partial realization of the principles, and six case studies that lacked the characteristics of the principles of productive disciplinary engagement.

Among the empirical studies on which Engle (2011) focuses includes Stein, Engle, Smith, & Hughes, (2008) related to five teacher practices for facilitating discussions. This paper focuses attention on the principle of problematizing primarily, as well as the other principles to a lesser degree. According to Engle, utilization of the five practices for orchestrating discussions engenders students' commitment to address the tasks while maintaining the intended cognitive demand of the task through a set of teaching strategies that include anticipating student responses, monitoring student responses that arise, selecting which student solutions to present to the class, sequencing the responses with a particular learning goal in mind, and asking preplanned questions that encourage students to connect the responses to each other in order to surface disciplinary ideas and practices. The implementation of the five practices described

above depends on the teachers' choice of tasks having a high cognitive demand (Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996; Stein, et al., 2000). Largely through the choice of the task and the instructional practices related to orchestrating classroom discussion, students are likely to experience uncertainty; a key element of problematizing. According to Engle (2011, p. 7), that is uncertainty about what to do, uncertainty about what to conclude, uncertainty about how to justify what one is doing or concluding, or uncertainty caused by competing alternatives about any of the three other issues. In addition, authority in terms of authorship and sense of intellectual agency is supported through student presentations to their class, in the five practices model. Through the practice of connecting, students are held accountable for relating to others the ways that disciplinary ideas are related. Selecting particularly innovative or canonical approaches to solutions through the process of selecting offers students the resources for disciplinary engagement as well as a way for the teacher to hold them accountable to the discipline and other students.

More recently, an entire section of the *International Journal of Educational Research* (2014, vol. 64) was devoted to reporting empirical studies related to the principles of productive disciplinary engagement. The journal included an introduction to the special issue authored by Forman, Engle, Venturini, & Ford, and four articles that shared in common the use of the productive disciplinary engagement framework. The article by Mortimer and Oliveira de Araujo, (2014) describes a private high school classroom for middle class students. Whether or not the teacher's practices can be considered instances of productive disciplinary engagement and whether teaching that encourages productive disciplinary engagement can co-exist with more traditional instruction were the foci of the study. Venturini and Amade-Escot's (2014) article describes a French middle-school classroom in an impoverished area where student safety

is a concern. This study focuses on the nature of productive disciplinary engagement in this challenging setting and the dynamic balance between the selected tasks and the resources provided. Meyer's (2014) study highlights how supporting productive disciplinary engagement may need to be considered as an iterative process, with necessary support changing from the initiation to later teacher experiences. The introduction to the special section, frames the articles and highlights the contributions of the articles to the productive disciplinary framework, as well as to engage the readers in considering ways of extending, challenging, and elaborating on the framework.

2.8 SUMMARY

The studies reviewed inform the proposed study in several ways. First, the aggregate of studies that individually offer consistent support of the principles described by Engle & Conant (2002) begin to validate the generality of their assertion; that is the principles of productive disciplinary engagement apply across learning environments in ways that could inform the design of new learning environments as well as research about existing ones. In other words, the composite of studies suggests that Engle & Conant's four principles of productive disciplinary engagement (authority, accountability, problematizing and using resources) may be used as a framework for analyzing features of the environment that contribute to student engagement. Studies that are largely focused on instructional practices related to whole group discussion support and define the four principles suggested by Engle & Conant (2002). The norms, structures, and classroom features that combine to create an environment supportive of productive disciplinary engagement encourages students to participate with increasing intensity in the practices of the community,

becoming active in the learning process (Boaler & Staples, 2008; Tarr et. al., 2008; Kazemi & Stipek, 2001; Chapin & O'Connor, 2007; Goos, 2004; Hiebert & Wearne, 1993; Silver, Smith & Nelson, 1995).

The proposed study contributes to the mathematics education field in several ways. First, it examines the instructional environment, teacher and student behaviors when the principles of productive disciplinary engagement are used as a tool to design the learning environment. If one agrees that the principles are present in effective learning environments, as I claim after reviewing the research related to whole group discussion, then using these same principles to design the environment seems to be a worthwhile endeavor. Instead of examining the environment after it has been established, results from this research might inform practitioners in ways that may enable them to construct the environment using the principles of productive disciplinary engagement as a design tool. Recognizing the complexity of integrating all four principles of productive disciplinary engagement simultaneously, it is my intent for the proposed study to include these principles while relating teacher practices and student behaviors. Understanding the nature of student behaviors when the principles of productive disciplinary engagement are used for the design of the learning environment will be essential toward understanding student learning. The guiding assumption is that for most students, the extent of their engagement in personal thought and the thinking of peers defines their learning. Learning to talk with peers regarding the discipline is critical, and depends on specific teacher behaviors to encourage this kind of behavior. So examining student behaviors will provide insight toward understanding student learning. Schoenfeld (2012) examined the complexities related to constructing a classroom analysis scheme and he proposed several well-grounded frameworks to establish a relationship between classroom practices and the student understandings that result

from those practices. His work causes reflection related to ways to capture the relationship between classroom practices and the robust student understandings that may be related to those practices.

3.0 METHODOLOGY

3.1 INTRODUCTION

This investigation explored the ways in which the four *principles of productive disciplinary engagement* may be used as a tool for informing the design of the norms, structures, and classroom features that combine to form a learning environment. The study examined both the instructional practices employed by the teacher and the nature of student engagement in a seventh grade classroom over the course of one unit of study, following the implementation of intentional pedagogical practices aimed at implementing the four *principles of productive disciplinary engagement* during the initial half of the year.

The study examined the following research questions using mixed methods. Given the goal of understanding the ways that the principles of productive disciplinary engagement are evident in the instructional practices implemented by the teacher and enacted by the student, data were collected to inform my understanding of the instructional practices and classroom interactions, students' views of their mathematics class, and the work required of the teacher to implement these principles in the classroom. Recognizing that qualitative research can never capture "reality" and that what is captured is really people's construction of the way they understand the world, I attempted to address internal validity through the use of the triangulation of multiple data sources (Patton, 2002). Data sources included video transcripts of one unit of

study, the tasks used within the unit, the student work that was produced as part of assessment or used in whole group discussion, teacher reflections of the lessons, and a student survey. Findings from these multiple sources were analyzed independently then in relation to one another in order to illuminate trends and themes across sources and to afford the opportunity to triangulate the data. By using multiple data sources that provide information related to one research question the validity of the results was established.

Validity was also established within the analysis of video transcriptions. Using complete lesson transcriptions enabled me to look for multiple instances of the principles of productive disciplinary engagement, as described in the research literature, within and across lessons as well as instances where those principles may not be apparent; establishing validity. Pattern matching (Yin, 2009) is described as comparing empirically based patterns with predicted ones. Patterns in this case are ways of enacting the principles of productive disciplinary engagement. In this study, internal validity was strengthened when research-based findings, reported in the literature, coincide with the empirical evidence in the classroom.

Research Question #1:

In what ways are the principles of productive disciplinary engagement: 1) evident in the instructional practices implemented by the teacher and 2) enacted by the students?

- A) In what ways does the teacher expand or constrict the distribution of authority within the classroom? In what ways do the students act with authority?
- B) In what ways does the teacher hold students accountable to themselves, peers, and the discipline? In what ways do the students engage in the social and intellectual practices that reflect accountability?
- C) In what ways does the teacher encourage students to take up intellectual problems

that simultaneously: engender genuine uncertainty in students, are responsive to learners' own interests and goals, and embody some central aspects of the discipline in question, that defines problematizing? In what ways do the students reflect genuine uncertainty in the instructional environment?

D) In what ways does the teacher encourage students to amplify their capacity to solve problems through the provision of resources? In what ways do students utilize resources to problem solve?

Research Question #2

A) What work is required of the teacher in order to translate the principles of productive disciplinary engagement into practice?

B) What challenges and successes does the teacher encounter along the way?

I chose a case-study approach because the purposes of the study demand an in-depth understanding of the situation and its meaning for those involved (Merriam, 1988). Because I was interested in the process of describing the elements of an instructional environment, it was necessary to describe interactions among the students, as well as their interactions with the teacher. The shared disciplinary norms, ways of interacting and roles assumed by participants needed to be thoroughly described.

3.2 CONTEXT OF THE CLASSROOM STUDY

This study focused on the students and teacher in one seventh- grade mathematics classroom. Each of the lessons that together comprise one mathematics unit from the seventh grade

Connected Mathematics unit focused on probability was captured. I chose this unit of study because students generally have limited experience with this topic, but have engaged in the study of proportional reasoning, a related unit, earlier in the school year. Students' limited experience with the topic of probability offered the opportunity to document the ways that students were productive in making connections to prior experiences with proportional reasoning and in developing generalizations related to ways to calculate probability without the interference of the prior procedural instruction. The unit of study included fifteen lessons and took place over a four-week period. The unit began with eleven sequential lessons, was interrupted by state testing, then resumed for four days the following week. In order to provide additional context for the study, the participants and the school are described in the following sections.

3.2.1 School

The current study focused on one class of students and one teacher in a seventh-grade regular education classroom at Berry Middle School over a unit of instruction related to the topic of probability. Berry Middle School, located in a suburban school district, includes approximately 620 seventh grade and 600 eighth grade students and was named a 2012 Blue Ribbon School and a 2011 and 2014 National and State School to Watch. Mathematics for seventh grade students is taught by five, regular education mathematics teachers and five special education teachers using a team concept.

Not including students with severe disabilities, seventh grade students in this district are enrolled in one of three seventh-grade courses based on standardized test scores, academic grades, and teacher recommendations. Using this criteria, the lowest-performing, quartile (approximately) of the students are enrolled in Pre-Algebra that includes three extra periods of

mathematics instruction per week; the middle two quartiles of the students are enrolled in Pre-Algebra which meets daily, and the highest performing students (approximately one quartile) are enrolled in Honors Algebra. The teacher/researcher who is the focus of this study taught four sections of Pre-Algebra that met for one period, five days per week and one section of Honors Algebra. Approximately eight students within this teacher's Pre-Algebra classes (90 students total) required modifications to instruction or assessments based on the demands set forth in the students' Individualized Education Plan (IEP).

3.2.2 Participants

The teacher /researcher and the students in the seventh period, Pre-Algebra class were the subject of this investigation. The nineteen students in the class included nine boys and ten girls. All twenty students were Caucasian. Students represented a range of socio-economic conditions that included approximately 10% of students who qualified for free and reduced lunches. Four students had Individualized Education Plans (IEPs). I chose to focus on the seventh period class for practical reasons related to the availability of a videographer. All four Pre-Algebra classes were a heterogeneous mix of students who performed relatively equally (between classes).

The researcher was a doctoral candidate in Mathematics education who served the role of teacher simultaneously. My background is atypical of mathematics teachers because my education and professional experience includes a B.S. in Materials Engineering and work in a related industry. Eleven years of teaching experience in grades K-4 preceded this teaching opportunity in seventh grade which I requested following a sabbatical year at the University of Pittsburgh in pursuit of a doctoral degree in Mathematics Education.

3.2.3 Instructional practices

The data for this study were gathered in a classroom following twenty two weeks of instruction that focused on purposefully implementing the principles of productive disciplinary engagement into the instructional environment. Although it would be impossible to characterize every element that was a part of the planning for this project, some of the decisions that influenced the character of the instructional environment are listed below.

- The principle of problematizing was addressed through the selection of tasks that were intended to develop students' conceptual understanding. Although the district uses the traditional Prentice-Hall Pre-Algebra curriculum materials, I chose to use the Connected Mathematics series as the basis of instruction, recognizing that these materials were designed to support teachers who were interested in teaching in a way that was consistent with the principles promoted by the NCTM and the CCSS-M. In addition to the Connected Mathematics lessons, in some instances I chose complimentary high-level tasks to include in lessons that I believed would support learning. (Tasks were gathered from NAEP release items, Mathematics education research, or tasks from the NCTM or Illuminations websites). Mathematics education research suggests a relationship between the task chosen and student learning (Boaler & Brodie, 2004; Stein & Lane, 1996). In addition, the principle of problematizing (Engle, 2011) draws attention to importance of the task in engendering uncertainty in the learner, and embodying "big ideas" of the discipline. Through the use of these materials, I have attempted to implement the principle of problematizing.
- The principle of accountability has been purposefully implemented using several pedagogical moves. Student assessments have included both individual paper/pencil

assessments as well as group assessments. This decision stems from my intent to illustrate that I value and take seriously the work that students do related to classroom tasks and to hold them accountable to classmates, the discipline, and to themselves (Forman & Ford, 2014). My perception of the necessary elements that combine to produce excellent quality work was defined in terms of rubrics for each assessment. Communication is an essential part of the environment. Students consistently addressed learning in groups of 2, 3, or 4; working together toward solution of the task and toward the production of a representation (or several) of the solution. The expectation for students to ask questions of each other, perform think-alouds, challenge thinking and reasoning of others has become the norm, and provides a social condition that invites students to revise their ideas. Explaining their ideas in light of others' ideas has also been promoted through the teacher's use of Accountable Talk. Fostering disciplinary engagement through purposefully encouraging students to consider how their ideas do or do not make sense is a central feature of the pedagogy that supports the principle of accountability.

- Developing student authority undergirds much of the structure of classroom lessons. Students work in small groups most of each class period. A whole group wrap-up usually follows wherein ideas are shared and discussed. As students work together in small groups, they develop solution strategies and make connections to other mathematical topics or representations. These differences in thinking then are aired during whole group discussions. Often, as students agree and disagree with solution methods presented, students exhibit passion about their view. As authors of the ideas, they exhibit the ownership that the principle of authority represents. In addition to

authoring ideas orally, students frequently create representations of their thinking in written form. These products offer the opportunity for students to evaluate other students' work and cooperatively develop rubrics that illustrate elements of exemplary work. In addition, often students participate in a "gallery walk" wherein they ask questions of student-author work using sticky notes posted directly on the product. These notes then provide opportunities for class discussion the following day, after I have had the opportunity to review them and identify partial understandings or ideas that deserve further discussion. In this way I try to balance authority with accountability. Homework assignments encompass several days; it is often assigned on Monday and due for submission on Friday. In this way, students have the authority to choose when to complete it, based on their own schedules. When it is submitted, a cover sheet must be attached. The cover sheet includes questions that the student must reflect upon including: what mathematical questions remain, what resources they used in completing the assignment, and names and phone numbers of peer resources they used. Homework is not graded for correctness because many parents participate in the completion of homework assignments while some students have no one to help them. Rather, students are awarded a small number of points for on-time submission that reflects "considerable effort".

- Resources such as manipulatives, virtual manipulatives using technology, graph paper, drawing tools, and peer groups are carefully considered in lesson planning. Using a variety of mathematical representations offers students ideational resources. Representations that include tables, graphs, student-drawn pictures, as well as other student-derived solutions offer peers a way to consider the task. Resources are

available for students to use at any time. In many cases, students have been reluctant to use the resources available; requiring encouragement and instruction regarding ways to use them.

The building Principal has supported my use of lesson plans that are not consistent with the district lesson plan format. The lesson plan format that I submit includes a section for each of the four principles of productive disciplinary engagement so that I have a tool to help me plan the ways that I will include these principles in every lesson. An example is attached in Appendix E. Pre-planned questions, anticipated student solution and errors, ways I will structure the class to offer students authority and provide resources for example, were planned in advance as much as possible.

3.3 DATA SOURCES

The examination of an instructional environment required a description of an amalgam of pedagogical features and student behaviors. Answering the research questions required a close examination of the classroom so as to determine the ways that students and the teacher interacted to construct the environment. With regard to research question one, there were several data sources that were used: verbatim transcripts of video recorded lessons, mathematical tasks, and student work that was produced as a result of the completion of assessments or was used during whole group instruction. In addition, a student questionnaire served to triangulate the data gathered by the teacher/researcher. In order to answer research question two, data sources included lesson plans, teacher reflections, mathematical tasks, and transcripts of video recorded lessons.

3.3.1 Video of instruction

Video records of daily, seventh-period, classroom lessons, transcribed verbatim, in their entirety, were the primary source of data for this study. Each forty-two minute lesson was videotaped using two cameras. One camera was set on a tripod to capture the activity of the whole classroom, while the other was hand-held and focused on one small group per class. For the purpose of this study, the video that captured the entire class was used. Only fourteen of the fifteen videos were useable due to a technical difficulty during one videotaping session. All students in the classroom had parent permission to participate in the study. Table 3.1 offers an overview of the lessons that comprised the study. Mathematical tasks featured in each lesson are detailed in Appendix I.

Table 2. Overview of Probability Lessons in the Study

Class	Focus Question/Topic	Tasks utilized during each lesson	Data collected
1	How does collecting more data help you predict the outcome of a situation?	Problem 1.1 ¹ (For this problem, students were also asked to graph the percent of heads versus the number of tosses, following a coin-flipping task.)	Video and student graphs
2	How does modeling with an experiment help you determine possible outcomes and the likelihood of each outcome?	Problem 1.2; part A,B	Video
3	How do you determine the relative frequency of an outcome?	Problem 1.3 A,B, C and Problem 6A	Video
4	Partner Quiz	4 questions and #19, p. 20 (#19 had been assigned as homework.)	Student written work Video of discussion related to #19.
5	Using evidence to support the analysis of events	Marble Task (Silver, Smith, & Nelson, 1995)	Written work of partners and video
6	Wrap up of marble task	Marble task continued	Video
7	Developing probability models: using a tree diagram to analyze outcomes.	Problem 2.3 and the Cafeteria Problem.	Video and student tree diagram
8	Using strategies to find theoretical probabilities.	Marbles task (NAEP released item) and the Sticky Gum Problem (Silver, Smith, & Nelson, 1995)	Video and student written work.
9	Using an area model to analyze compound events.	Problem 4.1A and 4.1C	Video
10	Using an area model to analyze compound events	Problem 1-4, pg 80 Making Purple	Video
11	Assessment	NAEP released item	Video and individual student work
12	Using an area model to analyze compound events	Problem 4.2D	Video (technical difficulty- no audio)
13 and 14	Simulating a probability situation. How is an area model for the one-and-one free-throw situation like or unlike the area model for the Making Purple game?	Problem 4.3 B ¹ and Problem 15, the Caves Paths task	Video
15	Assessment and student questionnaire	Three questions and the questionnaire.	Individual student written work and student questionnaire

¹Adapted from Connected Mathematics (Lappan et.al, 2014)

²Unless noted, the tasks were used without modification, from CONNECTED MATHEMATICS 3 WHAT DO YOU EXPECT? Copyright © 2014 by Michigan State University. G.Lappan. E.Phillips, J. Fey, and S.Friel. Used with permission of Pearson Education, Inc, 2014).

3.3.2 Mathematical tasks and student work

Another source of data that was used in the study included the tasks posed to students. Because the tasks chosen contribute to the establishment of problematizing within the instructional environment, they were considered carefully. Each task that was assigned to students was collected for later analysis. Student work was not coded, but was collected and used to provide clarity related to whole class discussions and assessments.

3.3.3 Student questionnaire

A student questionnaire, shown in Appendix A, that sought to gather student perceptions related to the classroom instructional practices and the classroom environment was given to students at the conclusion of the data collection process. I chose this time frame so that the instructional practices of this unit were those that were most recently experienced by students. The survey included questions related to all four principles of productive disciplinary engagement (authority, accountability, problematizing, and resources). Table 4.6 in the Results section highlights which questions address each principle.

3.3.4 Lesson plans

Lesson plans followed the format shown in Appendix E. As I composed the plans for this unit, I used the materials for the teacher in the Connected Mathematics (Lappan et.al, 2014) curriculum as a guide. Lesson plans were completed at least five successive days at a time, and modified as necessary, based on student thinking. As the unit commenced, lesson reflections allowed for a

comparison of the intended lessons and the enacted lessons. I added that reflective narrative to the bottom of each lesson plan. The format was consistent throughout the school year and encouraged the consideration of ways to include the four principles of productive disciplinary engagement during the lesson enactment.

3.3.5 Teacher reflection

In an effort to answer the second question regarding the challenges and successes of implementing the principles of productive disciplinary engagement, I kept a daily reflection journal wherein I captured my frustrations and barriers to implementation, as well as successes and surprises related to implementation. Reflections took the form of a journal entry that was completed immediately following the class. It was important that the reflection take place directly following the class because my perceptions of the class, the ways that students engaged and my own challenges and decision-making process during the class were best remembered then. Moment-to-moment decisions were affected by my knowledge and disposition. It was my intent to decompose my practice, and in so doing, to further our understanding of the character of this particular endeavor of implementing the principles of productive disciplinary engagement. Lesson plans and teacher reflections are found in Appendices J through V.

3.4 CODING AND ANALYSIS OF DATA

3.4.1 Data, Coding and Analysis: Research Question 1

A description of the plan for coding and analysis of the data follows. The description is separated into sections that detail coding and analysis plans related to each research question. For each question, each phase of analysis is described.

Research Question #1:

In what ways are the principles of productive disciplinary engagement: 1) evident in the instructional practices implemented by the teacher and 2) enacted by the students?

- A) In what ways does the teacher expand or constrict the distribution of authority within the classroom? In what ways do the students act with authority?
- B) In what ways does the teacher hold students accountable to themselves, peers, and the discipline? In what ways do the students engage in the social and intellectual practices that reflect accountability?
- C) In what ways does the teacher encourage students to take up intellectual problems that simultaneously: engender genuine uncertainty in students, and embody some central aspects of the discipline in question, that defines problematizing? In what ways do the students reflect genuine uncertainty in the instructional environment?
- D) In what ways does the teacher encourage students to amplify their capacity to solve problems through the provision of resources? In what ways do students utilize resources to problem solve?

The four principles of productive disciplinary engagement described by Engle & Conant (2002), include authority, accountability, problematizing, and resources. The ways in which these

principles were embodied in the learning environment were captured via transcriptions of videotaped classroom lessons, mathematical tasks, and student work that was associated with classroom discussions or assessments, described earlier. The coding of the video transcriptions with regard to each of the four principles was accomplished in several phases. A description of the collection, analysis, and reporting process, related to the ways that the teacher and the students reflect each of the four principles, follows.

3.4.1.1 Video transcripts

Engle & Conant (2002) define engagement using three criteria related to students. First, the number of students participating is indicative of engagement. That is, more students participating, and few students “off task” is considered as more engagement. Second, greater intensity in the way students participate in the mathematics instruction is greater engagement. Such intensity might be apparent as students’ speech overlap and the way they attend to each other with eye gaze and body position. Third, the extent to which participation of learners is responsive to others indicates greater engagement. Examples of responsive behaviors might include students making emotional displays, building on the thinking of others, and attending to their work for long periods of time. Further, Engle & Conant (2002) define engagement to be disciplinary when there is “some contact between what students are doing and the issues and practices of a discipline’s discourse” (p. 402). They define the word productive to include, “significant disciplinary progress from the beginning to the end” of students’ engagement (Engle, 2007, p. 215). They believe that productivity largely depends on the discipline, the task, the topic, and where students are when they begin to address the problem. Productivity, then, can only be judged on an individual basis. Disciplinary progress could be related to a design, making a new connection between ideas, or students shifting from explaining their own ideas toward a

posture that allows them to compare and challenge others' ideas. Authority reflects the idea that in order for students to become genuinely engaged in problems, they must have intellectual authority to do so. As learners are authorized to share their thinking, they become recognized as authors of the ideas and contributors to the ideas of others, leading to students becoming local authorities on a subject. In order to balance authority, accountability addresses the need for students to be accountable to explain their own thinking; making sense in light of other people's ideas. As accountability increases, learners improve their ideas so they are ready to be challenged more thoroughly by peers, internal authorities, and finally external disciplinary authorities (Engle, 2011).

As the teacher relinquishes authority students develop into classroom experts, producing knowledge with ownership. Students assume the task of developing problem solving strategies with increasing independence and also monitor the quality of their own work and the work of their peers. Examples of ways that a teacher may share authority with students may include encouraging a student to share a solution method, setting the expectation that students will ask questions of peers so that classroom discourse is not between the teacher and individual students, and redirecting student questions to other students.

Accountability in a classroom may include many facets including assessment practices and procedures, practices related to homework, norms related to the use of classroom tools, and other norms and procedures within a classroom. For the purpose of this study, I refer to accountability in terms of the interaction between individuals; the teacher and students, or among students.

The ways in which the environment embodies the principle of problematizing demands the analysis of both teacher and student behaviors. With regard to students, problematizing

includes examining ways that genuine uncertainty is evident. Engle points to four kinds of uncertainty that may be apparent: 1) uncertainty about what to do, 2) uncertainty about what to conclude, 3) uncertainty about how to justify what one is doing or concluding, and 4) uncertainty caused by competing alternatives about any of the three prior issues. Student behaviors that are indicative of uncertainty in the environment may include reassessing a solution path, trying a new tact, thinking but not writing, asking questions related to what to do or what to conclude, multiple attempts to justify a position orally or in writing, to name a few. Problematicizing also suggests specific actions and choices made by the teacher. In order for problematicizing to be present in the environment the task chosen must create genuine uncertainty, be responsive to the learners' own interests and goals, and embody some central aspects of the discipline in question (Engle, 2011). In addition, the definition I use for problematicizing includes the selection of a task that meets the criteria of one of high cognitive demand (Stein, Smith, Henningsen, & Silver, 2000). These criteria suggest that the task chosen by the teacher is of significant importance to ensuring that problematicizing is embodied in the environment.

The provision of relational, material, and ideational resources amplify a student's capacity to participate in productive disciplinary engagement. Video footage of classroom episodes in both small, collaborative groups and whole group discussion, provide the data to determine the ways in which resources are evident in the instructional environment. Relational resources as described by Nasir & Cooks (2009) include the relationship between students and each other, and students and the teacher. Material resources may include physical manipulatives, the time to work on a problem, or physical mathematical tools. Resources that offer the student a way to think about a task or their posture about a task would be termed an ideational resource. An alternate representation, an organizing method for data, the development of language that

could be used in expressing mathematical ideas or the teacher's emotional stance would all be examples of ideational resources.

The first phase of the plan for data coding included transcribing each video in its entirety and then highlighting every instance of the principles of productive disciplinary engagement that were apparent in each of the video transcriptions using the operational definitions defined prior to the start of the investigation to guide my work. Instances of the four principles of productive disciplinary engagement were highlighted in the original transcripts in contrasting colors. Review of the transcript and modifications to the operational definition occurred iteratively until each transcript had been reviewed exhaustively. The operational definitions that were used as a starting point for identification in the transcripts follow.

Operational definition of shared authority:

- Students demonstrate authority when they make claims and anticipate the potential critique of others. Student authority will be apparent in the way students explain their mathematical reasoning when they offer potential solutions. Careful consideration of the reasons behind their solution indicate students anticipating the critique of others. For example, "I think three is the most likely outcome, because there are four chances of getting three and only two chances for the other numbers."
- Students simultaneously make ideas their own as they critique the reasoning of others. Critiquing the reasoning of others may be apparent in both oral and written student work. A position-driven discussion offers students the opportunity to change their mind. They may initially agree with one side of the argument, then reconsider their position; based on the critique of the reasoning behind their position. An hypothetical example of such a discussion follows.

Teacher: Henriët, do you agree with Bob or Madia?

Henriët: I understand Bob's reasoning, but Madia's also is making sense.

Ute: I disagree with Madia because there is a higher proportion of yellow faces than green faces.

Henriët: Oh, I see Ute's point. There must be a larger chance of rolling a yellow because the number of yellow faces to the total number of faces is a higher percent of the total.

- Students initiate ideas that are taken up by the teacher and class. As the teacher or another student gives credit to a student for an idea, and the class then engages in a discussion of that idea, authority is being shared.
- The teacher positions herself as a peer through the use of first-person or inclusive pronouns such as "this tells me.." (Cornelius & Herrenkohl, 2004).
- Turn taking is I-R-R-R (Cazden, 2001; Mehan, 1979).
- The teacher's use of revoicing acts to promote student power (O'Connor & Michaels, 1993, 1996).
- The teacher uses language that places herself as a co-learner. In so doing she shifts the power from herself to her students because she has placed them in a position to decide what part of the message to believe or adopt (Cornelius & Herrenkohl, 2004).
- The students are allowed and encouraged to question each others' thinking and theories (Boaler & Brodie, 2004)

Operational definition of accountability:

- Learners are positioned to evaluate their own ideas and those of others (Davies & Harre, 1999; Yamakawa, Forman, & Ansell, 2009).

- The teacher consistently presses for explanations that include the underlying mathematical thinking; not just procedural steps (Kazemi & Stipek, 2001) .
- Students are encouraged to agree /disagree with classmates using mathematical reasoning and justification in sustained exchanges.
- The teacher and students respond with respectful, substantive interest when another student genuinely shares what he is really thinking about a topic.
- The students are encouraged to justify their reasoning to themselves during written work; what O'Connor & Michaels (1993) would call being accountable to themselves.
- The students are encouraged to justify their reasoning to peers during small group work and whole class discussions. The teacher frequently accomplishes this task using Accountable Talk moves listed in Appendix F (O'Connor & Michaels, 1993, 1996), and questions characterized by Boaler & Brodie, (2004), as “probing”, “generating discussion”, “linking and applying”, and “exploring mathematical meaning” (Appendix B) .

Operational definition of problematizing:

- The task chosen is one that is high-level (Appendix C) according to the Math Task Analysis Guide (Stein, Smith, Henningsen, & Silver, 2000), and represents “big ideas” of mathematics for students of the grade level.
- The teacher has chosen a task that engenders genuine uncertainty in students that may include: uncertainty about what to do or what solution path to follow, what to conclude, how to justify what one is doing or concluding, or caused by competing

alternatives about any of the three prior issues. Students must exert effort to arrive at a task solution and cannot immediately solve it.

- The students do not give up and ask the teacher for the solution to a challenging task; they struggle toward the common goal of solution.
- The students express uncertainty in the form of statements of uncertainty, asking questions, rereading the question/problem, sitting silently, seeking material resources.

Operational definition of resources:

- Resources include material, ideational, and relational resources (Nasir & Cooks, 2009). This definition expands that of Engle & Conant (2002).
- Material resources include traditional tools such as manipulatives, paper, spinners, clocks, or any other tool that amplifies a student's capacity to problem solve.
- Relational resources include other humans and human interaction, such as that of peers, coaches, parents, teachers.
- Ideational resources help students construct meaning through linking of ideas such as through the use of multiple mathematical representations (tables, graphs, pictures), or ideas for questions that students may ask of peers in the form of a wall-mounted chart. Material resources will be limited to paper, pencils, manipulatives, while ideational resources will include mathematical representations, or alternate ways of illustrating a problem or solution.
- Student errors or partially solved problems are used to encourage students to construct meaning.

- Time is considered as a resource by the teacher. Adequate time is provided to students to solve problems. Students have time to complete the task with care, but not so much time that they engage in off-task talk or activities.

As I highlighted each transcription for the four principles, using the operational definitions above, patterns related to the ways each of the principles was apparent began to emerge. For example, as I highlighted instances of students assuming authority, a pattern related to the teacher positioning herself as a peer became apparent. Additionally, other patterns became apparent, such as students' noticing different features of tasks and solutions. Students repeatedly exhibited specific behaviors. As I saw behavior patterns emerging, I used sticky notes to index instances of each. Piles of sticky notes, related to certain teacher or student behaviors informed the coding scheme. The coding scheme that emerged from the use of the operational definitions and that was ultimately applied uniformly to all transcripts is shown in Appendix G.

Video records also provided the necessary data to produce event maps of each class. An example of an event map is illustrated in Figure 3.1. The event maps provide a macro-level view of each class, allowing the reader to locate the transcribed examples within the lesson. The value of this capacity is that the reader may glance at the event map and recognize what events and structures have come before or after the segment being described in the text; providing perspective for the reader. For example, the event map shown in Table 3.1 illustrates for readers the changes from small group to whole group instruction as well as the time spent on individual topics, called topically related segments (TRS). This information helps the reader to recognize that in this lesson, the class' attention to developing a generalization related to the theoretical and experimental probability included a computer simulation for the whole group, followed by whole group discussion of the salient points of the simulation. Next small groups discussed the

generalization, followed by a whole group wrap up. The event map provides a “birds-eye” view of the lesson. When placed in parallel with transcript selections throughout this document, that include time stamps and line numbers, the event map offers a way for the reader to accurately locate the event within the context of the lesson.

Each event map was created following the transcription of a lesson. Because the topically related segments necessitate the identification of occasions within the lesson where the class is focused on one topic, I watched the videotape repeatedly, concurrently reading the transcript. When the topic changed, and the class was refocused, I marked the transcript to indicate the change. In addition, in order to produce the event map, I needed to identify whether students were working in small or whole group. Changes in the activity structure and the topic are located on each event map, including the elapsed time with the lesson.

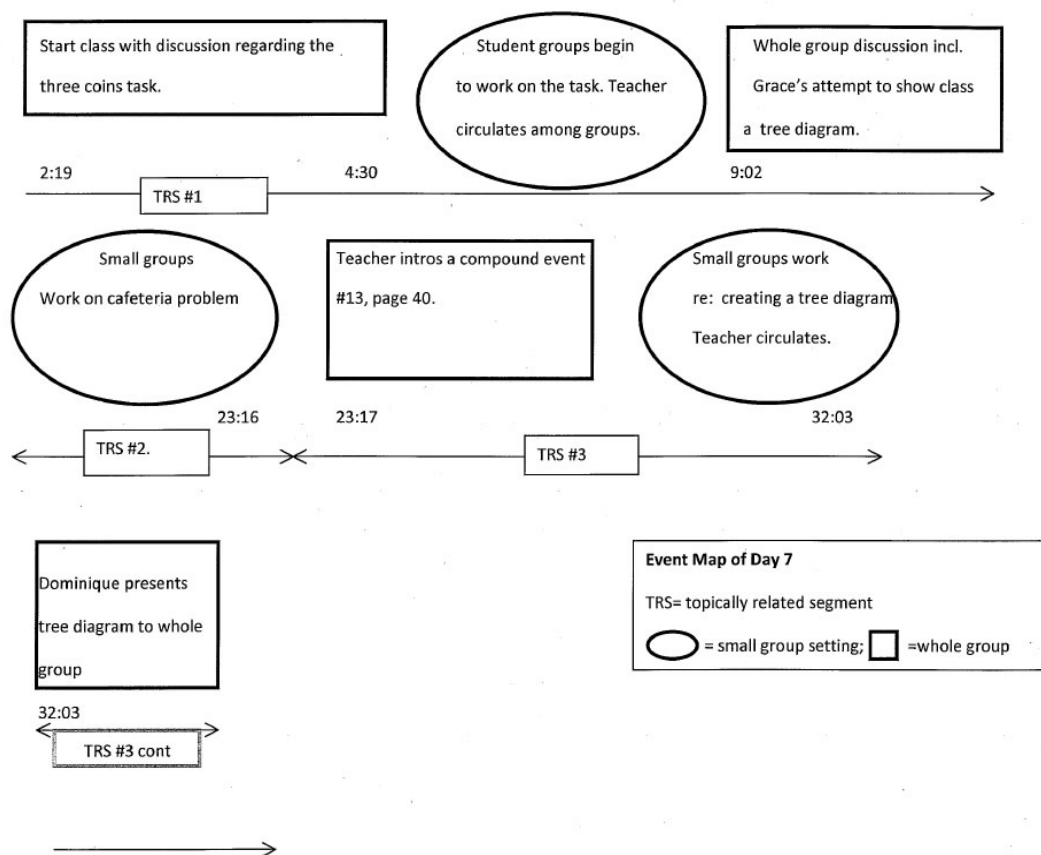


Figure 1. Sample Event Map

3.4.1.2 Mathematical tasks

The tasks posed, that contribute to the establishment of problematizing within the instructional environment, were coded using the Math Task Analysis Guide (Stein, Smith, Henningsen, Silver, 2000) shown in Appendix C. The tasks were carefully selected to: 1) meet the criteria of a task of high cognitive demand, 2) engender genuine uncertainty within students, and 3) reflect mathematics that is part of the CCSS-M content standards for seventh grade. Because tasks labeled “doing mathematics” in the Math Task Analysis guide require complex, non-algorithmic thinking, do not suggest a solution pathway, and require students to self-monitor their progress, these tasks often produce anxiety or uncertainty in students. Through the analysis of the

potential cognitive demand of the task, I characterized the tasks that have been assigned. Individual tasks used in lessons were scored. If more than one task was used in a lesson each task was scored. According to the definition of task referenced previously, individual questions were not scored. Rather, a group of questions designed around one concept were scored. For example, during the first day's work, students attended to Problem 1.1 in the Connected Mathematics materials (Appendix I). The task had three parts, labeled A-C, and several questions within part B. I coded the entire Problem 1.1 with one designation using the Math Task Analysis Guide. Task scores were reported as a percent of tasks used for each category of the Math Task Analysis guide. For example, "73% of tasks used in this unit fell within the Procedures with Connections category." This information provided insight regarding instructional elements that contributed to enacting the principle of problematizing in a classroom, including the instructional task.

According to the Math Task Analysis Guide, tasks with lower level cognitive demands are labeled either memorization tasks or procedures without connections tasks. Both of these groups of tasks are focused on producing the correct answer and have little ambiguity about what needs to be done and the way to do it. An example of such a task used in this study is homework problem 1, assigned day 2. The problem states, "Mikki tosses a coin 50 times, and the coin shows heads 28 times. What fraction of the 50 tosses is heads? What percent is this?" Based on the idea that most seventh grade students already understand a procedure for finding a percent of a total, this task is algorithmic.

With regard to probability, an example of a task deemed *procedures with connections* is homework question three, assigned on day 2 of the study. The problem says, "Kalvin tosses a coin five days in a row and gets tails every time. Do you think there is something wrong with the

coin?” Although the solution depends on a procedure, there is no solution path that is prescribed and the solution requires some degree of cognitive effort.

At the highest level of cognitive demand are *doing mathematics* tasks (Stein, Silver, Smith, & Henningsen, 2000). An example of such a task is the well-documented task shown in Appendix I, day 5, the Marble task (Silver, Smith, & Nelson, 1995). This task offers students multiple solution paths, and the exploration of relationships among mathematical concepts. For seventh graders, students must draw on their knowledge of proportional relations, graphing, and linear relations and develop connections to probability. Because there is no algorithm presented, students must explore and develop a solution path themselves.

Table 3. Student Work Codes

Code	Definition	Example	Principle
US	Student uncertainty	“I don’t understand”	Problematizing
JS	Student justifying reasoning without prompt	“I know the answer is 2 outcomes because...”	Accountability
N	Learning to Notice- student sorts through information provided and selects features of the task to use in solution.	“I notice that the 36 is 3 times the size of 12.”	Authority

3.4.1.3 Student questionnaire

A student questionnaire (Appendix A) offered students the opportunity to express their view of the ways that they participate and engage in the principles of productive disciplinary engagement. The first portion of the survey was used to inform the results of research question #1, while the second portion informed the results of research question #2. Table 4.6 informs the

reader regarding which principle is addressed by each student question. Frequency of student responses for each category was tabulated for every question. This questionnaire served as one method of triangulation of data regarding the ways that the principles of productive disciplinary engagement were present in the instructional practices and enacted by students. Student perceptions of the mathematics classroom norms and instructional practices that were a part of the student questionnaire were compared with the descriptions of the classroom environment as presented by the teacher/researcher. Comparison of the data gathered from each perspective provides validity to claims. For example, the data may reveal, according to the researcher, that students often demonstrate authority through the independent decision to use the document projector to illustrate their ideas. A question in the student questionnaire to determine their perception of the use of the document projector seeks to understand their view. Internal validity will be strengthened if both the researcher and students agree on that particular point.

3.4.2 Data coding and analysis: Research question 2

Research Question #2

- A) What work is required of the teacher in order to translate the principles of productive disciplinary engagement into practice?
- B) What challenges and successes does the teacher encounter along the way?

Fostering norms that support a learning environment conducive of productive work by all its members has been explained using the concept of “the principles of productive disciplinary engagement” (Engle & Conant, 2002). These principles, designed for use in research, are repurposed in this study as guiding principles in the creation of the learning environment. They are being subjected to the conditions of practice and examined in a concrete situation.

Recognizing that, “the knowledge that is used to analyze teaching is not entirely the same as the knowledge that is used to teach” (Lampert, 1990, p. 37), the research question posed here points to the examination of the teacher’s role, the specific instructional strategies the teacher uses, and the practical reasoning entailed in teaching lessons having a specific character in one particular setting.

3.4.2.1 Lesson plans

In order to answer research question 2, I examined lesson plans in relation to the lesson reflections, looking for consistencies and inconsistencies related to the plan versus the way the lesson actually unfolded. In addition, I examined the lesson plans alone for features of the plans that were common to all or most of the plans. Through the examination of the plans and reflections, I was able to identify critical components of the planning process and suggest reasons for inconsistencies in the planning/enactment of the lessons that contribute to understanding the work of the teacher.

3.4.2.2 Student questionnaire

The results of the second portion of the student questionnaire contributed information related to the establishment of social practices in the classroom. The question, “How often does this happen in your mathematics lessons?” offered students the opportunity to express their views of some of the norms established. The quantitative results of the survey were compared to results gathered through the analysis of the transcripts to establish validity. Quantitative results included tabulating the frequency of each student response category for every question.

3.4.2.3 Video tapes and transcriptions

Because the enactment of the principles of productive disciplinary engagement depend on robust discussion, with the teacher positioning herself as one who is interested in students' mathematical thinking, verbatim transcriptions were used to consider what teacher moves contributed or inhibited the interpretation of students' mathematical thinking as each lesson proceeded. Examining the transcripts for teacher moves that emerged within and across lessons provided information related to the work of the teacher at specific moments within lessons. Specifically, I examined the transcripts for evidence of the use of five practices for orchestrating productive discussion, a tool that I have purposefully incorporated into lessons for many years (Stein, Engle, Smith, & Hughes, 2008).

Another way that the transcripts were used was to examine teacher questioning. The Boaler and Brodie (2004) questioning framework was utilized to examine the work of teacher questioning. The importance of teacher questions in shaping the nature and flow of classroom discussion has been identified as a critical and challenging part of a teacher's work (Boaler & Brodie, 2004; Hiebert & Wearne, 1993). The Boaler & Brodie (2004) study resulted in a tool that was useful for categorizing teacher questions, used in this study (Appendix B). Sharing the definition of a question, used by Boaler & Brodie, every teacher question in fourteen lessons were coded. (The twelfth class included a technical difficulty, wherein no audio was captured.). I chose to include utterances that had both the form and function of a question. That is, I excluded statements that sounded like a question but didn't function as such. For example, "Would you like to share your thinking?" would not be coded for it serves as an imperative. In addition, questions needed to be mathematical in nature. I excluded questions like, "Do you

have your homework?’. If a question were repeated exactly, I counted it only once. Results of the coding were reported both in tabular and graphical form and described in the text.

3.5 REPORTING

3.5.1 Research question one

The results of analysis related to research question one will be in the form of rich, thick, descriptions that uses verbatim transcripts, mathematical tasks, student work, and the results of the student questionnaire as evidence of the ways that the principles of productive disciplinary engagement were 1) evident in the instructional practices implemented by the teacher and 2) enacted by the students. The patterns that emerged as a result of the coding process were also reported. Event maps of each class were reported as well.

The mathematical tasks chosen were scored using the Math Task Analysis framework (Appendix C). Tasks were scored and reported individually as well as a percentage of the total in each task category. A display that included the lesson day number, the mathematical task, its cognitive demand, and the lesson goal provided a way to examine general trends regarding the task selection on a day-by-day basis and across days.

The resources utilized, were reported in tabular and descriptive form for each instructional day. Reporting in tabular form allowed for the consideration of general trends across lessons and the consideration of the resources that aligned with each task presented.

The student questionnaire offered insight into the students’ perceptions of classroom instruction and the environment. Results of this data source were compared with that of the

transcriptions of classroom videos. Questionnaire results related to the first question, “How much do you agree with the following statements about your teacher in your Math class?” were reported both quantitatively and qualitatively. Responses related to the second question in the student questionnaire are reported in the Results section related to research question #2.

The results of video analysis and transcription were reported using specific examples from multiple lessons that shared common features. For example, multiple examples from several lessons wherein problematizing could be identified using the cognitive demand of the task combined with the expression of student uncertainty were reported using descriptive language combined with excerpts of verbatim transcription, and task code. Verbatim transcriptions were reported in tabular form for longer sections and embedded in the text for sections comprising less than ten turns of talk.

3.5.2 Research question two

The work that is required of the teacher in order to translate the principles of productive disciplinary engagement into practice as well as to identify the challenges and successes that she encounters along the way were reported in the form of thick, rich description; making every attempt to connect lesson plans, decision making, lesson reflections, and classroom events continuously and transparently. Patterns and themes that might inform others who are interested in implementing the principles of productive disciplinary engagement were reported.

The student questionnaire, question number two, addresses students’ perceptions regarding many of the social practices that contribute to the definition of the classroom norms. Who assumes responsibility for certain things, who explains and who listens at what times, who is free to move about the room and who is not, are just a few of the social practices of which

students are acutely aware. The frequency of student responses on the questionnaire, were tabulated and reported. These responses shed additional information regarding the work of the teacher.

Data regarding the challenges and successes of the teacher reflect my thought process. I examined lesson reflections and considered challenges and successes on a day-by-day basis and also across the unit that I encountered.

3.6 INTERRATER RELIABILITY

In an effort to establish interrater reliability, a second coder was trained in the use of the abovementioned coding schemes using a sample transcript with accompanying video, and samples of student work. Specifically, three complete, randomly selected lesson transcripts (approximately 20% of the data) were double coded using the coding schemes in Appendix G (my framework related to the Principles of Productive Disciplinary Engagement), and Appendix B (Boaler & Brodie (2004) framework). In addition, all of the mathematical tasks used throughout the data gathering period were double coded using the Math Task Analysis guide (Stein, Smith, Henningsen, & Silver, 2000). The second coder was a doctoral student, familiar with Mathematics Education and Learning Sciences research. The training included a review of the coding schemes and independent coding of a sample lesson segment. Coding results were compared and differences were resolved by consensus. An agreement level of approximately 95% was achieved. Following all coding, the results were reviewed with Dr. Margaret Smith, co-author of the Math Task Analysis Guide. Her insights regarding codes for several tasks were considered and disagreements were resolved through discussion, resulting in full agreement.

3.7 SUMMARY

This study explores the ways in which the four principles of productive disciplinary engagement may be used as a tool for informing the design of the norms, structures, and classroom features that combine to form a learning environment. The study examines both the instructional practices employed by the teacher and the nature of student engagement in a seventh grade classroom over the course of one unit of study, following the implementation of intentional pedagogical practices aimed at implementing the four principles of productive disciplinary engagement during the initial half of the year. The data sources included video of classroom lessons, tasks in which the students engage, event maps, the student questionnaire, and student work related to assessments or whole class discussion for research question one, and lesson plans, lesson reflections, the student questionnaire, and video of classroom lessons for research question two. Data sources were analyzed using mixed methods. Reporting of the results of the analysis was in the form of thick, rich descriptions of the findings, tables, and figures, as well as through the quantitative results of the student questionnaire and the teacher questions.

4.0 RESULTS

The results of the data analysis reported in this chapter are organized into two sections corresponding to the two research questions presented in Chapter 1. Each section has subsections that explain patterns related to the principles of productive disciplinary engagement that became apparent as a result of the data analysis process. As a reminder, the research questions are shown below.

Research Question #1:

In what ways are the principles of productive disciplinary engagement: 1) evident in the instructional practices implemented by the teacher and 2) enacted by the students?

- A) In what ways does the teacher expand or constrict the distribution of authority within the classroom? In what ways do the students act with authority?
- B) In what ways does the teacher hold students accountable to themselves, peers, and the discipline? In what ways do the students engage in the social and intellectual practices that reflect accountability?
- C) In what ways does the teacher encourage students to take up intellectual problems that simultaneously: engender genuine uncertainty in students, and embody some central aspects of the discipline in question, that defines problematizing? In what ways do the students reflect genuine uncertainty in the instructional environment?

- D) In what ways does the teacher encourage students to amplify their capacity to solve problems through the provision of resources? In what ways do students utilize resources to problem solve?

Research Question #2

- A) What work is required of the teacher in order to translate the principles of productive disciplinary engagement into practice?
- B) What challenges and successes does the teacher encounter along the way?

These results include those from the analysis of verbatim transcriptions of classroom lessons, students' assessments, the mathematical tasks used in each lesson, a student questionnaire, and event maps of each lesson. Consistent with the research questions, Section 4.1.1 addresses the ways that the teacher and students enact the principle of authority. Section 4.1.2 addresses the ways that the teacher holds students accountable to themselves, peers, and the discipline as well as the ways that the students engage in the social and intellectual practices that reflect accountability. The principle of problematizing is addressed in Section 4.1.3; addressing the mathematical tasks selected by the teacher and the uncertainty expressed by the students. The tasks and uncertainty are examined both independently and as well as in the ways they relate to one another. Section 4.1.4 addresses the provision of resources by the teacher, as well as the ways that students utilize resources to problem solve. Each of these sections includes sub-sections that describe patterns that became apparent during the analysis process. As coding proceeded, certain student behaviors or teacher practices began to repeat themselves. The coding process thus informed the reporting of results. In each of the sub-sections I have included topically related segments that serve to illuminate certain features of the lesson. I have chosen to use multiple examples, highlighting some features, while backgrounding others. The reason I

have made this choice is to make it clear to the reader that the lesson features that represent these principles did not occur one time, but rather were embedded in every lesson. As each principle is considered, the features that I wish to highlight are discussed beginning with gross features, and moving toward more subtle features that are apparent in the data. Short sections of transcriptions are embedded in the text itself, while longer classroom segments are included in figures. The results include exemplars drawn from every lesson in an effort to demonstrate that the principles of productive disciplinary engagement were apparent across the data gathering period, not just in one or two isolated lessons. The specific lesson day that provides evidence for the patterns identified is noted for each transcription. In addition, both time-stamps and line numbers are noted to afford the reader quick reference to the lesson portion under consideration. Event maps of each lesson allow the reader to get an overall sense of each lesson and to place the topically related segment, that might be the focus of the discussion, into perspective related to the events that preceded or followed the highlighted segment. An overview of the way each lesson was utilized in the results is shown below. The column entitled “Patterns Illustrated” refers to section titles of this document. Because I am both the researcher and the teacher, I have chosen to refer to myself as “I” when I am in the role of the researcher and “the teacher” when I have assumed the role of the teacher.

Table 4. Patterns Illustrated During Each Lesson

Lesson/day number	Patterns Illustrated
Day 7 and 2	Participation pattern, turns of talk
Day 8, 2, 13	Teacher as partner and offering choices
Day 3, 7	Critiquing the reasoning of peers
Day 7 and question 3 (assessment)	Opportunity to notice
1	Positioning students as authors
4,5,6	Placing students in a position to publically revise
9,10	Demonstrating intellectual courage to hold peers accountable
3,5	Student uncertainty
5 , 14	Resources
4	Planning, enactment, reflection

4.1 RESULTS RELATED TO RESEARCH QUESTION 1

4.1.1 Ways that the teacher and students enact the principle of authority

Engle and Conant (2002) define authority with regard to two ideas. The first idea is related to students having an agency in defining, addressing and resolving problems. The second includes members of the learning community positioning students as stakeholders by publicly identifying them with the claims, approaches, explanations, designs and other responses to problems. Students may develop into classroom experts to whom others rely for help. Students who have authority are encouraged to be authors and producers of knowledge rather than consumers of it. In other words, students become active learners who take responsibility for their own learning (Hufferd-Ackles, Fuson, & Sherin, 2004). Enacting this principle demands that teachers share

authority with students in developing the learning community, and in so doing provide the opportunity for students to develop a sense of agency.

In order to enact the principle of authority, my goal for students was to learn not only about the ways of solving problems related to probability, but also that the warrant for doing so comes from a mathematical argument, not from a teacher or a book. Consequently, my intent included teaching mathematical content, technical skills and knowledge of the discipline, simultaneously with teaching the way to participate in the disciplinary discourse of the class while using all the resources available to them. Central to the process was positioning students as capable, independent, decision-makers who had much to offer each other and the class. It was my goal for students to attribute success to their own action; to consider themselves as being responsible for their own learning. The “I can DO this...I can figure this out” attitude means that students consider themselves to be capable of acting strategically when they encounter an unfamiliar task, believing that they can be successful, and knowing that their ideas are of value. Generating ideas and strategies must be something that is valued and students must recognize that they are in control of the process of generation. That is dichotomous with a student thinking, “My answer was right because the teacher helped me.” This contrast suggests a change in the role of the teacher from one who is explicit in the way that she thinks about a type of problem, then expects the students to duplicate her thinking in the solution of like problems. Expanding the distribution of authority places students in an active versus a passive role.

The subtitles in this section reflect the patterns that emerged as part of the coding process. For example because every code included an indication regarding whether it was indicative of a student or teacher response, the participation pattern became apparent. Likewise, I coded for instances of the teacher as partner stance and times that students were offered choices. Only

after the coding was partially complete did I notice that these two codes emerged together. Therefore, in the results section that follows, they are reported in the same section. These patterns, taken together, were used to develop a holistic view of the classroom norms, structures, and classroom features that combined to form a supportive environment for students in the enactment of the principles of productive disciplinary engagement.

4.1.1.1 The participant pattern

The environment on which this study relies was created to purposefully develop a participation pattern wherein students were active participants. Encouraging active participation by students with the goal of creating an environment where all voices may be heard demands attention to changing the relationships of power within the classroom (Cornelius & Herrenkohl, 2004). Power to assess information and monitor progress; traditionally held by the teacher must be assumed in part, by students. One of the ways that power is apparent is through the examination of who does the talking in the class. As students assume the ownership of ideas, the expression of ideas becomes both a right and a responsibility.

The analysis of the verbatim transcriptions of the fifteen classes that were part of this study indicates that active student participation was accomplished. Every lesson includes examples of students assuming authority through atypical participation patterns. The typical IRE (Mehan, 1979) participant structure was replaced by students assuming consecutive turns of talk. This participation pattern is apparent in the topically related lesson segment shown in Table 4.2. In the seventh lesson, the teacher has selected Estelle to come to the document projector to describe her tree diagram, representing the following problem. “Suppose that you spin the pointer of a spinner at the right (having 2 colors) once and roll the number cube. The numbers on the cube are 1,2,3,4,5,6. Make a tree diagram of the possible outcomes resulting from a spin of

the spinner and a roll of the six-sided number cube.” (Lappan, et.al, 2014). The diagram that Estelle presented is shown below in Figure 4.1. A portion of the classroom discussion that followed Estelle’s presentation is shown in Table 4.2. This segment will be examined in more detail in subsequent sections, but currently I draw attention to only the feature of turns of talk.

Table 5. Topically Related Segment Related to Estelle’s Tree Diagram

	Teacher	Student	Function/Commentary
(323)	Teacher: Estelle, you want to share your tree? (32:03)		The teacher is asking her to make her thinking public; holding her accountable.
		Estelle: I’m not 100% sure.	Student expresses uncertainty.
(338)	Teacher: That’s ok. Throw it up there. And let’s see. My computer went blank. I might have to put my password in again. Ok folks. Take a look up here. Is this what yours looks like?		The teacher asks for intellectual courage and for the student to explain how they understand. She is sharing authority with both Estelle and the class. The question begs students to compare features of their own work with that of Estelle.
		(lots of student comments at the same time)	

Table 5 (continued)

(353)	Teacher: Let's think about it . I'm sure she'd be happy to take feedback. She said right off that she wasn't that sure. But she's being a good sport about it.		Using a teacher as partner stance, the teacher relays that the reasoning process will be public and cooperative. She makes clear that it is ok to be unsure and that it's safe to present incomplete understanding.
		Nancy: I agree. It goes one through six. It goes 1,2 then goes 6 on a side. I agree with that one, just not so much the one below	Demonstrating her authority to evaluate Estelle's tree diagram.
(357)		Estelle: I'm not sure how I thought of the dice.	Again expresses uncertainty and implies that she can't explain her logic but she understands that it is expected that she do so.
(362)	Teacher: Use a pen (on the paper under the document projector)		The teacher holds her accountable suggesting that she remain there and write with a pen that can be clearly seen by the class.
(371)		Estelle: For the dice you only roll 1 through 6, so that's' like 1,2,3,4,5,6. Then if you roll one of them. Like if you roll a 1 or 2 after thatI guess that is what I was thinking.	She responds to the indirect message to attempt to explain her thinking.

Table 5 (continued)

(383)	<p>Teacher: What are you thinking, folks? (long pause- students do not talk)</p> <p>So what are your outcomes, Estelle? I'm not sure what to think of the outcomes.</p>		<p>The teacher asks a question to generate discussion. When students do not begin to talk, she directs the conversation to Estelle.</p> <p>Teacher takes a teacher-as-partner stance and in so doing asks the student to explain her thinking on a specific topic. She inserts vocabulary into the question.</p>
(366)		<p>Estelle: Let's just take this one for example. If you spin a 1, you can get 1,1,; 1,2, 1,3, 1,4 , 1, 5, 1 ,6 like that's only if you spin a 1 on the spinner the die.</p>	<p>Makes her thinking public, using the tree diagram under the document projector as a resource.</p>
(391)	<p>Teacher: Ok..</p>		<p>She relays that she is listening carefully.</p>
		<p>Estelle: Then, for like the dice, if you rolled a 1,2,3,4,5, or 6, then if you spin the spinner, you can either get like a 6,1 or 6,2...</p>	<p>Continues to make her thinking public with regard to the second tree diagram.</p>

Table 5 (continued)

(400)	Teacher: So what do you notice about these, guys? What's the same or different about the two...the one that starts with the spinner or the one that is the dice. Just talk to her.		(Exploring mathematical meanings and/or relationships and encouraging discussion) She encourages students to participate without hand-raising and "just talk" using a discussion format.
(409)		Inez: I think they are the same, they just have a different base. You started one with a spinner and one with the dice.	She sees a relationship among the two representations and introduces the word "started" ; implying that Inez is visualizing actually spinning the spinner and rolling the dice. She demonstrates her authority to determine the mathematical merit of each representation.
	Teacher: K.... Dennis, does yours look like that?		Holds Dennis accountable for listening and participating.
(423)		Dennis: It doesn't look like that but I agree with that.	Responds to request.
	Teacher: Does she need both of them?		(Generating discussion and exploring mathematical meaning/relationships)
		Several students say "no" together.	Using authority to assess mathematical merit.

Table 5 (continued)

(431)		Nya: I'd say the top one is less confusing. The bottom one looks like a lot of lines.	Uses authority to assess mathematical merit.
	Teacher: Estelle, I'm a little confused. I'm thinking ...does the student need all those outcomes or do you have two representations of the same thing? How many outcomes are you representing there? I'm not sure.		Using the teacher as partner stance, models thinking aloud. She expresses confusion and simultaneously interjects vocabulary and requests her to share logical reasoning. (inserting terminology and probing)
(446)		Estelle: Honestly, I think both represent the same thing.	Expresses intellectual honesty implying that there is good reason to change it.
		Bob: Why are they like that? Like they're two separate things.	Expresses that he hasn't followed the logic presented.
	Teacher: This one has both the spinner and the dice in it. Right? (writing at the doc projector). These are the dice. Right?		Teacher assumes authority and begins to explain at the document projector.
(560)		Bob: I see that.	He implies that the discussion is not beginning in the right place. He already understands that.

Table 5 (continued)

		Nancy: She just did the dice backwards from the spinner. (inaudible) If you just picked one it would be the same as the other one.	She tries to restate the logic presented earlier; assuming authority without any prompting from the teacher or students.
(572)	Teacher: What's your question, Bob?		Teacher tries to determine the point of partial understanding for Bob.
		Bob: Why are they linked?	He expresses confusion about having 2 tree diagrams.
(578)	Teacher: They're not linked . They're 2 separate things. The way she has them written, she wrote them 2 ways but they are the same thing. They represent the same number of outcomes. This is 12 and this is 12. It's the same 12 outcomes written in 2 different ways. You don't see it.		Teacher again assumes authority to clarify the meaning of the representations that Estelle presented.

Table 5 (continued)

(595)		Lyla: It's like she said before. It's like if Estelle started with the spinner then went on to the dice or and the second one she started with the dice then went on to the spinner.	She assumes authority and restates the logic behind each representation.
		Nancy: It's(overlapping)	She begins to do the same.
		Bob: Ohhhhh	Bob expresses that he now understands the logic.
(610)		Ute: It's like the die you end up with any number 1 thru 6 but the second one is the other way around. Say it lands on one. It says the spinner could land either 1 or 2. They are the same exact thing just in a different order.	Ute summarizes the discussion, assuming authority with no prompting.

Throughout the segment, the non-traditional role of the teacher and students is apparent. In contrast to a traditional classroom where the pattern follows the ubiquitous initiation-response-evaluation (IRE) form (Mehan, 1979), the pattern takes on a different complexion. The teacher neither speaks every other turn, nor does her talk take on an authoritarian tone. The teacher turns in this exchange follow the pattern shown: (T=teacher, S=student) T-S-T-S-S-T-S-S-T-S-S-T-S-T-S-S-S-S. The turn-taking pattern itself indicates a change in power, with students consuming more of the talk turns. Similarly in the following segment shown in Table 4.2, as students discuss homework problem number four, during the second day of the probability unit, the teacher–student turns follow the pattern: T-S-T-S-S-S-S. Homework problem number four was, “Len tosses a coin three times. The coin shows heads every time. What are the chances the coin shows tails on the next toss? Explain.” (Lappan, et.al, 2014).

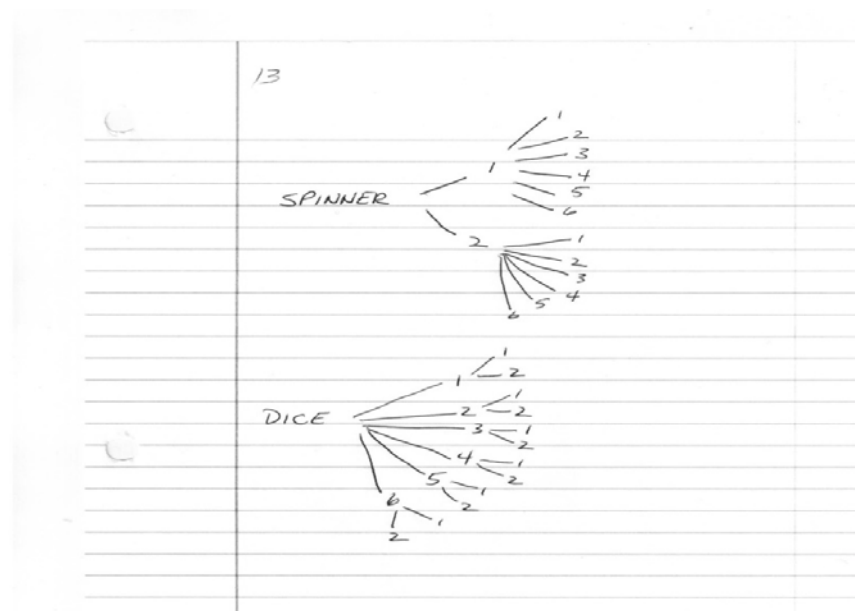


Figure 2. Estelle's Tree Diagram

Table 6. Example of Participation Pattern, Turns of Talk, Day 2

	Teacher	Student	Function/Commentary
50	Teacher: Are there any ones you want to talk about? I have one I want to talk about but are there any that you guys want to talk about? (pause- no one replies) Ok. So let's talk about number 4 for just a couple minutes. (Teacher walks to the whiteboard) (5:25) The problem says that the person flipped four heads (writing H, H, H, H, in a vertical line on the board), and wants to know what the probability is for the fifth one. So you don't need to raise your hands, just talk. Henriët, you want to start?		The teacher is asking students to assume authority by choosing a homework problem they'd like to discuss. When there are no suggestions from the students, the teacher makes a selection.
71		Henriët: There are only 3 heads.	Student corrects the teacher's expression of the task.
	Teacher: Three heads. Forgive me. (Erases one of the H's on the board) So what do you think? Henriët, what do you think about this?		The teacher corrects here error.
80		Henriët: Well I said it would be one-fourth, cuz I (teacher writing $\frac{1}{4}$ on board) I said if he had already had head three times....i would think that the fourth time it is more likely he'd get a tail. (teacher sitting with a group near the board) (Pause)	Student provides her explanation.

Table 6 (continued)

90	Lyla: I think it's still like a 50% chance because even though it's heads three times, that doesn't really matter. Because no matter how many times you flip it, it's going to be a 50-50 chance because it's a fair game, it's equal chances. (Teacher still sitting)	Student disagrees with prior response and explains her position.
99	Bob: I have to agree with Lyla. I mean you're using the same coin. The chance is always going to be 50-50. I mean, there's a heads there's a tails (Bob, crouches on his seat) . There's only two chances. Two choices I mean.	Student assumes a position and explains his reason.
108	(6:56) Nancy: I agree. But can't you like have a coin that has both on one side?	Student assumes a position.

These examples typify student participation during whole group discussion. Many students are engaged in expressing their thoughts and ideas. However, not every student was equally engaged. Two students (Henry and Dennis) were both were very engaged in small group discussions but rarely spoke during whole group discussion at all. Henry spoke in whole group discussion when he was prompted to do so, but Dennis largely refused to share his thinking. A segment that highlights the way that Dennis participated is shown in Table 4.1 and discussed in the section related *to students holding each other accountable*.

As other features of the enactment of the four principles of productive disciplinary engagement are discussed in the remaining portions of this document, I will draw attention to additional examples of teacher-student turns of talk.

4.1.1.2 Offering choices and the teacher-as-partner stance

Offering choices to the students and the implementation of a teacher-as-partner stance (Tabak & Baumgartner, 2004) is discussed together, because as I coded the data, the two repeatedly occurred in pairs. Offering choices and the teacher-as-partner stance worked together to offer students power; making them decision-makers while backgrounding the authority of the teacher. Together, these two pedagogical features helped to create a level of symmetry in the social configuration of the classroom. Offering choices demands that the teacher recognize the multiple ways of approaching problems, making connections, bringing prior knowledge and experiences to the task, and the long term value of allowing students to grow through the process of decision-making. Recognizing that students' backgrounds have afforded varying degrees of experiences with the decision making process, offering choices to students that allows for decision-making practice becomes of even greater significance. The teacher-as-partner stance contributed to students assuming a decision-making role. As the teacher deflected the decision making to students, through her teacher-as-partner stance, they quickly assumed the responsibility.

As I examined the data for ways that positioning contributed to offering students authority, the task selected by the teacher surfaced as one way that she offered students an opportunity to make choices and practice decision-making skills. Because high-level tasks (Stein, Smith, Henningsen, & Silver, 2000), discussed in more detail in section 4.3, offer students a variety of solution possibilities, students made choices regarding where to begin and what solution method to use, depending on what method was accessible for them. This choice, often negotiated with a partner or several peers in a small group setting, positions students as capable, decision makers who must take an active role. Offering choices to students is in

contrast to a situation wherein one method of solution has been practiced by the class, and the next problem set is predictably more of the same.

An example discussion from the eighth class in this study follows.

(11:30) Teacher: This kind of problem makes it hard for me to keep track of my thinking.

Bob: Ya

Inez: I tried one of those tree things...it didn't work out.

Teacher: Is there anything else you could use? If the tree thing is being hard for you, is there any other thing you could use to keep track of your thinking?

There are two ways that authority is distributed in this excerpt. First, the teacher positions herself as someone who would have a similarly difficult time if she were engaged in the same intellectual work. When she says "...hard for me"... she seems to also be a person who is fallible and who has experienced similar struggles. She has positioned herself as a partner (Tabak & Baumgartner, 2004). When Inez admits that she had tried unsuccessfully to use a "tree thing" (tree diagram), the teacher suggests that there may be other options and that she knows that Inez is capable of making an alternate decision. They clearly agree on the value of record-keeping; the decision to be made is what method to use. The teacher leaves that choice to Inez. Had the task in which the students engaged been one that had one solution process; one way of addressing it, the preceding discussion would have been unlikely to occur. However, because the task offered choices to students, and the teacher deflected the decision-making process, Inez was placed in a position of authority.

Student choice and the teacher positioning herself as a peer is also apparent in the following topically related segment. In this lesson (day 2), the teacher is introducing a task wherein students will collect data resulting from flipping cups. As she introduces the task, she

reminds students that the theoretical probability is unknown to them, unlike the prior experience they had flipping coins. She offers students a choice in how to keep track of their data, as opposed to distributing a work sheet or table. However, she makes it clear that they are accountable for some record-keeping system. In addition, the use of the pronoun, “we” and “we’re” positions the teacher as a partner; someone who will be engaged in the same intellectual activity.

Teacher: “We’re going to do that today. We’re going to flip cups. (Nya raises her arms over her head.) You don’t know what the theoretical probability is. So I need you to flip the cup thirty times. And I need you to keep track, however you think you should keep track. We’ll combine our data and see what we get.” (Day 2) (adapted from Lappen et.al. 2014, p.10)

Also implied, is the idea that the teacher believes that the students can be successful in this assignment. The described task demands that they mentally engage in developing a plan for recording their data and prepare to share it with the class. The comment, “We’ll combine our data and see what we get” implies that there is a mystery to be solved and that students will need to take an integral role in solving it based on the information.

Additionally, students routinely made choices regarding where to physically position themselves. Because they were not confined to their seats during whole group discussions, students often walked to the white board or document projector to explain their thinking; freely moving about the room. The freedom to move about the room contributed to students assuming authority and positioning each other as experts. In the following segment from day 13, a student has been asked to share his thinking regarding an area model that represents a compound event; a one-and-one free throw. The task is shown in Figure 4.2.

In the district finals, Nishi's basketball team is 1 point behind with 2 seconds left. A player on the other team fouls Nishi. Now she is in a one-and-one free-throw situation. This means that Nishi will try one free throw. If she makes it, she tries a second free throw. If she misses the first free throw, she does not get to try a second free throw.

An area model representing the possible outcomes for Nishi's one-and-one free throws is shown below. Explain what you know about the design of the area model (This task was adapted to include a partially completed area model as below)

1. Why are the blocks the size they are?
2. How would you label the left side of the area model? Explain.
3. How would you label the top side of the area model? Explain

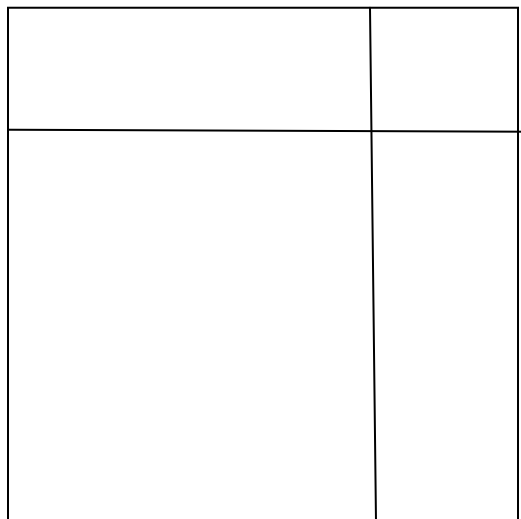


Figure 3. One-and-one free throw task and area model. Adapted from (Lappen, et.al., 2014, p. 76), Day 13

During the day 13 class the students engaged in a discussion of the task described in Figure 4.3. The area model that is part of that figure was reproduced onto the white board prior to the start of class. Students walked to the whiteboard freely as they commented on the model.

Following Ute's comment, (line 394), in the following exchange that summarizes his small group's work, the teacher reminds students of the classroom norms and encourages them to freely make contributions. Immediately following her statement, Bob walks to the board and engages Ute with his thinking. He is followed by Lyla, Sydney, and Estelle, who walk to the white board to engage Ute and the class in their thinking during day 13 class.

(line 394) Ute: We put yes and no for the chances of making it (pointing to the horizontal line at the top of the area model.) I think I have this messed up (begins to erase the 60% and 40%. But um, this side, because it's bigger is 60% of the (inaudible) and this 40. (Students have hands raised))

(23:00) Teacher: You don't have to raise your hand. You can talk, you can walk up there, talk to him. I don't know what you're waiting for. But there has to be ONE conversation.

Bob: (stands up and goes to Ute) What I'm thinking right now is can you make this chance to be given again. Like can you add the line up here.

Ute: (inaudible) (Bob sits down)

(Lyla goes up with encouragement from her group)

Lyla: With ours, we didn't draw a line here. We just counted this section and it was 36. So that's a 36 % chance she would make it on the first shot. The second one, that's like ..this would be 24 (writing in the box) . So that's the 24% chance she'll make it on the second shot.

Sydney: And that would equal 60%

Nya: That's how like we did it in the beginning.

Bob: (Begins to sing)

(Lyla writes $36+24=60\%$ on the board)

(There is discussion among students but not audible)
(Estelle walks to the board)

Teacher: Ok. Hold up. Keep watching.

Estelle: I guess. I didn't understand this until just now. This would kind of be like the first shot (pointing to the orange section) If she doesn't make it, she doesn't (inaudible), but this could be (pointing to the purple section) like (inaudible)

Teacher: Your thinking is really building on each other. I'm really thrilled about that. I want to add some labels. Nobody is putting a label on here. How about if this was Shot 1 (writes "shot 1 " on the left side of the model) Assume that's shot 1. Keep going Estelle. Now if that's shot 1 , how does that divide that box by putting shot 1 there.? What does the orange and purple mean if that's shot 1?

Interestingly, students have assumed authority; taking responsibility for actively immersing themselves in the details of the task and the thinking of their peers. Several students were willing to assume an intellectual risk, walk to the front of the room, and add to the discussion. The teacher has contributed little to the discussion; but has encouraged students to freely contribute. It is not until after Estelle speaks that the teacher interjects by placing a label on the area model. She applauds students' efforts to listen and respond to peers' ideas. Throughout the segment, students are positioned as independent, decision-makers by the teacher and each other. They eagerly respond to the opportunity to make choices and assume responsibility.

Three elements of student choice have been documented in the preceding discussion: choice as to the way to enter the task and represent their solutions to tasks, choice as to the way to keep track of their thinking during the solution process, and choice as to where to physically place themselves in the classroom. Choice as to the way to enter the task was unexpected on my part. Although everyone was asked to utilize an area model, students at their desks reproduced the area model on graph paper. Using the squares in the graph paper as a resource, Lyla counted the centimeter squares to gain entry to the task. When she talks about counting 36, she is talking about counting graph paper squares. With regard to keeping track of their thinking during the solution process, there was no prescribed method regarding the way to make sense and label the

area model. The necessary connection between the area model and the word problem belonged to the student, so the way they kept track of that connection was also their own. Some students used the words, “Make” and “Miss”, while others used only percentages as they worked.

Essential to each of these choices is the task itself. The task has contributed to student uncertainty; resulting in students having a genuine need to exchange ideas with one another or for the teacher to resolve uncertainty directly, herself. She has chosen to share authority with students, and encouraged them to author the solution using reasoned explanations. Integral to all the choices offered to students is the way the teacher has positioned students as independent, decision-makers who are capable of producing knowledge and authoring ideas.

4.1.1.3 Critiquing the reasoning of peers

Critiquing the reasoning of peers is an indicator of student authority. As students developed a sense of agency, they assumed some of the roles that are traditionally held by the teacher including the evaluation of ideas. In Table 4.2(Day 7), all of the bolded turns indicate student turns wherein the primary function of the talk was assessment of the information. Several students demonstrated intellectual courage and engaged in Estelle’s thinking. Nancy began the discussion of Estelle’s diagram saying, “ I agree. It goes one through six. It goes 1,2 then goes 6 on a side. I agree with that one, just not so much the one below.” In making this evaluation, she has necessarily immersed herself in Estelle’s diagram, truly attempting to analyze her thinking and make a connection between that and the diagram. She attempts to restate the logic presented earlier; assuming authority without prompting from either the teacher or students. Likewise, Lyla assumes authority and restates the logic without prompting. Later, Inez also critiques Estelle’s thinking when she says, “I think they are the same, they just have a different base. You started

one with a spinner and one with the dice.” Finally, Ute assumes authority to summarize the discussion. All of these students provide examples of student agency.

Common to the examples related to critiquing the reasoning of peers in the study were students having the time and agency to talk to each other about an idea or representation. Extended examples of students critiquing peer reasoning occurred during the wrap up of a task wherein students presented solutions. With regard to the discussion regarding Estelle’s tree diagram, Figure 4.3 illustrates the location of the discussion within the class. The topically related segment that addresses Estelle’s tree diagram begins about thirty-two minutes into the class; after small groups have had the opportunity to discuss the task.

As noted in the example with Estelle’s tree diagram, the selection of the solutions to be presented to the class was an essential element in encouraging other students to critique peer thinking (Stein, Engle, Smith, & Hughes, 2008). The selection of solutions to be presented during whole class discussions will be discussed again with regard to the work of the teacher. Estelle’s tree diagram offered the opportunity for the teacher to formatively assess the capacity of students to understand the tree diagram model. They thought abstractly in bringing meaning to both representations in her diagram.

The Day 3 class affords another example wherein students carefully consider the reasoning of peers. Students were addressing Problem 6A, page 17 from Connected Mathematics, What Do You Expect? (Lappan et.al, 2014). These tasks are shown below.

Problem 6A, page 17. (Lappan et.al, 2014)

Kalvin tosses a paper cup once per day each day for a year to determine his breakfast cereal. Use your results from Problem 1.2 to answer the following.

- How many times do you expect the cup to land on its side? On one of its ends?
- How many times do you expect Calvin to eat Cocoa Blast in a month? In a year? Explain.

Problem 1.2, page 11

A. Conduct an experiment to test your prediction about how a paper cup lands. Toss a paper cup 50 times. Make a table to record your data.
(Lappan, et.al. 2014)

Students had gathered data regarding (Problem 1.2), tossing cups, in the previous lesson.

Problem 6A asks students to use the proportions related to the times the cup lands on its side or end to scale up to the number of times it would occur in 365 tosses. The context provided is that Calvin gets to eat Cocoa Blasts each time the cup lands on its side.

The discussion provides another example of students critiquing the reasoning of peers. Following small groups addressing the problem, Nya is asked to come to the document projector to share her thinking.

(17:42) Teacher : How about you Nya?

Nya- I did mine wrong.

Teacher- Let's see it anyway.

Although the student expresses uncertainty, the teacher insists that she share her thinking with the group.

(206) Nya (walks to document projector) – Okay well I couldn't really find a number and I don't know why. But I also based it off of our own logic, by what we did in our groups. So in my group it landed pretty much on its side most of the time. So I expected it to... So pretty much it's an estimate. I expect mine to be about three out of four times. Just because of the way the cup shaped and how it's not like a coin.

Teacher – Is that what you got yesterday? Three out of four times it landed on its side?

Nya – a little bit more.

Teacher – OK, so you used that kind of as a round number?

Nya – I did mine as logic.

Teacher – Okay, so did you come up with a number?

Nya – no.

Teacher – Could you?

Nya – Probably.

Teacher – Can somebody help her? She says yesterday it landed on its side three out of four times. Is there a way to use that to figure out how many times out of 365? (hands go up) Just talk.

The teacher has recognized that Nya is considering the results regarding her cup flipping from the day prior proportionally. However, she has not transitioned that proportional understanding to the capacity to use scale.

(220) Sydney – I was kind of thinking you could divide 365 by 4 and then you do however many, like 3 sides – three times out of the four sides.
(Nya is writing at the document projector)

Teacher – So the 365, she is saying divide by 4 is $91 \times 3 = 273$.

Sydney – So it would be like 273 days out of the one year.

In this exchange, Sydney makes a suggestion for a way to consider scaling. The teacher restates her suggestion and adds the product of 91 times 3. She then encourages the group to talk without raising their hands.

(224) Teacher – Just talk.

Henriet- Is this ABOUT this much?

Teacher – She’s asking if this is exact or about. What do you think about this?

Henriet critiques the reasoning involved in the discussion; questioning whether the result would be an approximation or an exact answer.

(227) Henriet – I know it’s not exactly. Would you write “about”?

Teacher – Would you write “about”? Ute’s saying yes. Why would you write “about”?

This segment demonstrates the students actively engaged in the thinking of their peers. The assumptions regarding how exact the answer needed to be and what the answer would represent was the topic of discussion in this small segment. The teacher's moves to encourage discussion, emboldens peers to participate in directing the direction of the conversation.

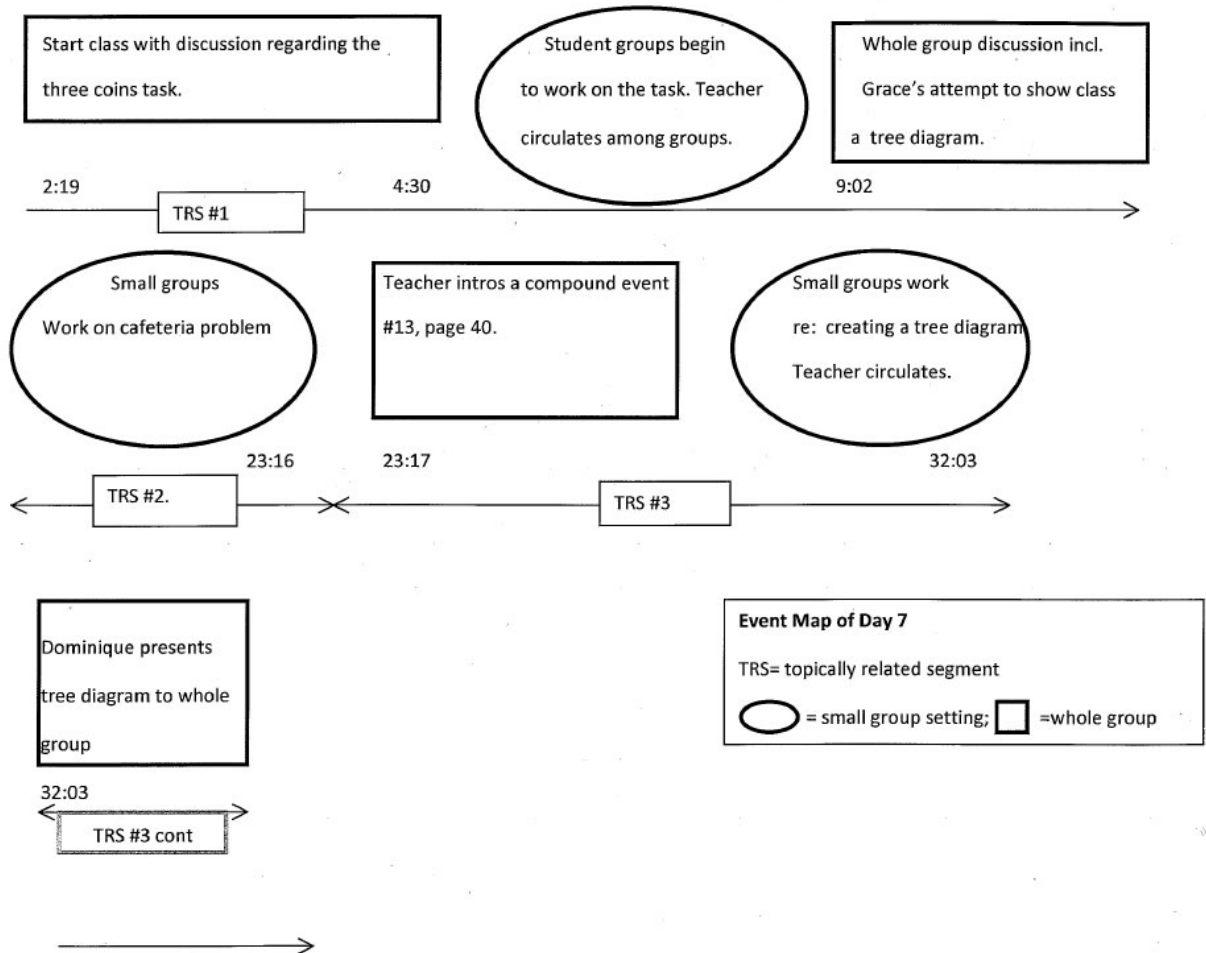


Figure 4. Event Map of Day 7 Class

4.1.1.4 The opportunity to notice

Among the codes developed during the analysis process was “noticing”. The distribution of authority was apparent in the particular way the teacher invited students to notice features of

mathematical information in whole group presentations. Students were encouraged to select a piece or pieces of information from competing information; what is often referred to in mathematical literature as noticing or, more specifically, executive attention (Hatano & Greeno, 1999; Loboto, Hohensee, & Rhodehamel, 2013). Through the invitation to notice, students necessarily were placed in a position as an active participant, challenged to sort through the visual cues; selecting and sorting information in an effort to identify particular mathematical features among competing bits of mathematical information. Features noticed by one student were not necessarily the same as the next student. What each student noticed served to draw attention to his own thought process, and to distinguish his thinking from another student's thinking. As each student shared, she donned an author's hat to provide an explanation in the form of a narrative. In sharing what she has noticed with the community of learners, she has authored a mathematical idea necessary of consideration by the members of the community. This noticing then served both an individual and a community function; as both the individual and other students sought to identify the pattern or feature as the original author viewed it. He immersed himself in the thinking of his peers. Several examples of such instances are illustrated in the discussion of Estelle's tree diagram, Table 4.2.

In the classroom discussion in Table 4.2, that begins thirty two minutes into the lesson, the teacher asks students to notice features of Estelle's tree diagram. She directs their thinking to compare the two and to "just talk to her". This phrase reminds the class of the established norm where students don't need to raise their hands to speak, but rather are expected to share their thinking via a conversation. Directly following the teacher request to notice, Inez provides her observation. She evaluates the two representations and determines that they are "the same..they just have a different base." In terms of noticing, Inez has examined the information, sorting and

selecting the necessary information. She determined that there was a relationship between the two representations and demonstrates her authority to determine the mathematical merit of each representation. Further, Inez has visualized actually spinning the spinner and rolling the dice when she says , “You just started one with the spinner and one with the dice.” Nya (in subsequent lines) also has sorted and selected information to discuss. She has compared the two representations and determined that the top one is “less confusing”. Nancy and Lyla also have compared the two representations, immersing themselves in Estelle’s thinking, and they have determined that the diagrams represent the same action of rolling the dice and spinning the spinner. Ute then summarizes the discussion in his own words. He says, “It’s like the die you end up with any number 1 through 6 but the second one is the other way around. Say it lands on one. It says the spinner could land either 1 or 2. They are the same exact thing just in a different order.”

While this exchange demonstrates students assuming authority as they actively construct meaning and express ideas, the teacher has also immersed herself in Estelle’s thinking and asks questions that serve to draw attention to specific features of Estelle’s representation. Taking a stance as teacher as partner (Tabak & Baumgarnter, 2004), mentioned earlier, she draws attention to the difference in the number of potential outcomes when she says, “Estelle, I’m a little confused. I’m thinking...does the student need all those outcomes or do you have two representations of the same thing? How many outcomes are you representing? I’m not sure.” Through the use of a probing question, she has invited Estelle to compose an authoritative narrative. Expressing confusion and simultaneously interjecting vocabulary, she is requesting Estelle to explain something that may have been an unconscious decision.

Although Estelle's representation was central to the discussion, the fact that Bob couldn't seem to notice the features that Inez, Nya, Estelle, and the teacher had discussed early in the conversation prompted additional conversation. Following Bob's question, "Why are they like that? Like they're two separate things", the teacher and Nancy try to determine the point of partial understanding for Bob. Following the teacher's attempt to explain at the document projector, where she assumes authority, Nancy restates the logic, assuming authority without a teacher prompt. Following another question by Bob, the teacher again tries to clarify the meaning of the representations. Her attempt is followed by Lyla who assumes authority, by also trying to bring clarity to Bob's thinking. Lyla's comment gives authority to Nancy's earlier comment when she says, "It's like she said before...." . Ute, quiet throughout the discussion, assumes authority by summarizing the discussion. The work of helping Bob to notice salient features of Estelle's representations was the work of the community; not solely owned by the teacher, and served to socialize the class' attention to mathematical features of her display.

Important to the discussion that invited students to notice the mathematical features presented by Estelle was the task itself and the way the lesson was enacted by the teacher. The task as it was presented offered students the opportunity to represent the mathematical ideas in more than one way. No one answer was sought by the teacher, nor was the answer viewed to be the most important element of the discussion. The task offered the community something worthy of discussion; something that would cause students to think and reason. The way the teacher enacted the task also provoked discussion. As students worked on this task in small groups, the teacher had monitored student work and noticed that Estelle had an unusual representation. Through deliberate selection of Estelle's work for whole class discussion, the teacher offered the class a representation that was likely to bring about a rich, mathematical

discussion, centered on student thinking and reasoning (Stein, Engle, Smith & Hughes, 2008). In terms of Lave & Wenger's (1991) idea of a community of practice, both the students and the teacher helped to apprentice those students who were less experienced at noticing to the key features of the activity and their significance.

Noticing was apparent in student written work as well and contributed to students' capacity to make a conjecture and provide an explanation; an act indicative of students' authoring ideas and making connections among ideas. Question 3 of the final assessment is shown in Figure 5.

3. Ann Marie has a spinner that is divided into four regions. She spins the spinner several times and records results in a table. Based on her results, make a drawing of what the spinner might look like.

Region	1	2	3	4
Number of Times Spinner Lands in That Region	9	4	12	11

Figure 5. Question 3- Final Assessment

This question demands that the student use proportional reasoning or part –to-whole relationships to predict the size of the spinner pieces. Several student responses point to the importance of identifying patterns in the question that allows them to develop a conjecture and provide a supporting explanation. Ute's response, shown in Figure 6 is one example.

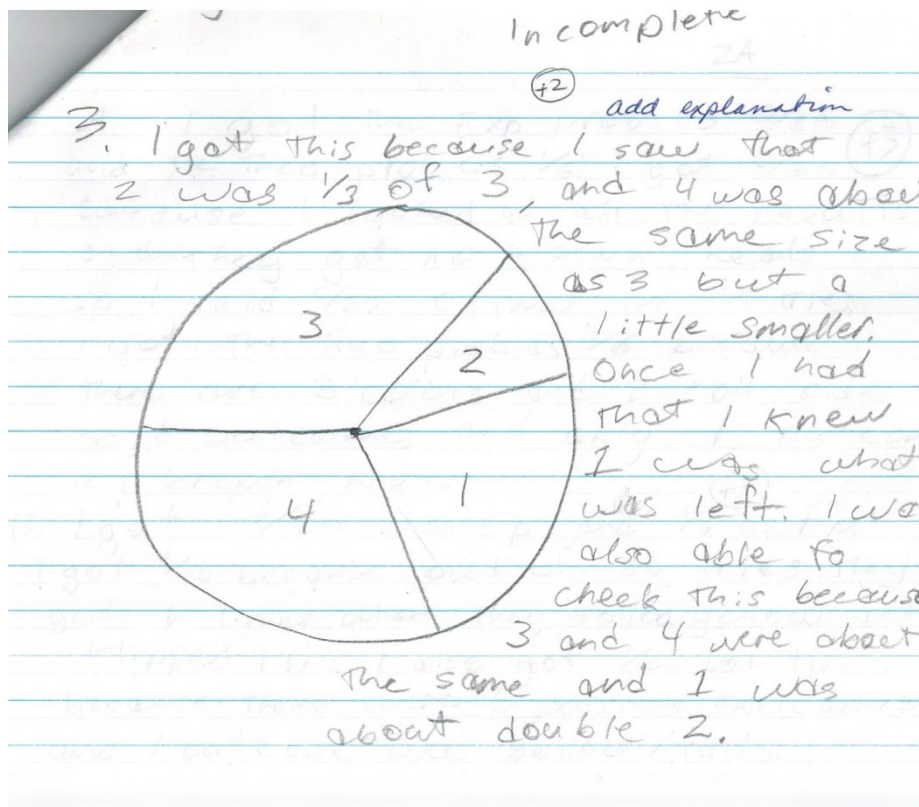


Figure 6. Ute's Response to Question 3

His response uses the word "saw" instead of "notice", but he notes explicitly what information he used in developing his conjecture. He says that he used the number of times the spinner landed in region 2 was one third the times it landed in region 3. He also noticed that the number of times it landed in region 1 was approximately double that of region 2. He then develops an explanation that reflects the information he has used from the problem statement. Interestingly, the teacher had returned the assessment to Ute because when he first submitted it, he had included little explanation.

Henriet also *noticed* features of the problem, but what she noticed was quite different than Ute's observation. Her response is shown in Figure 7.

I think that the spinner would look like this because I already
 knew that region 3 would be the biggest because it landed there
 the most. I know that she spun the spinner 36 times, 12
 which landed in region 3 - $36 \div 12 = 3$. Since the quotient
 3, I know that a third of the circle would be region 3.
 Since region 4 is 11, one less than 12, I know that just under
 $\frac{1}{3}$ of the spinner was region 4. Region 1 got 9 - $36 \div 9 = 4$
 so I knew that $\frac{1}{4}$ of the circle would be region 1. This leaves
 the remainder of the circle for region 2.

Figure 7. Henriët's Response to Question 3

Henriët has noticed the part-to-whole relationship of the number of times the spinner lands in a region. Using her observation, she chose to develop her conjecture using fractions to represent the relationship. She has noticed that the total number of spins was 36 and that 12 times, the spinner landed in region 3. She clearly recognizes that region 3 should represent $\frac{1}{3}$ of the circle. She relates each part to the whole value of 36. Her explanation makes the rationale supporting her conjecture easy to follow. Conversely, Inez's response indicates that what she *noticed* included a larger grain size. Inez's response shown in Figure 4.7, indicates that the feature that she noticed in the task was that there were four sections to the spinner and the number of times the spinner landed in each region was not the same. She used the information not to compare the numbers to each other using part-to-whole relations or proportional relations, but rather only with regard to relative size (e.g. 9 is larger than 4 so the spinner piece may be larger). Her response may have been more elegant had she observed that the number of times the spinner landed in a region could be considered in relation to each other or as parts of the total 36 spins. Only eight students of the class of nineteen *noticed* the relationship among the numbers in

the task that allowed for a reasoned conjecture between the number of times the spinner lands in a region, supported by logical explanation. The remainder of the students used the idea that larger numbers of times the spinner lands in a region correlates with a larger portion of the spinner. There was no mention of the exact size of the region.

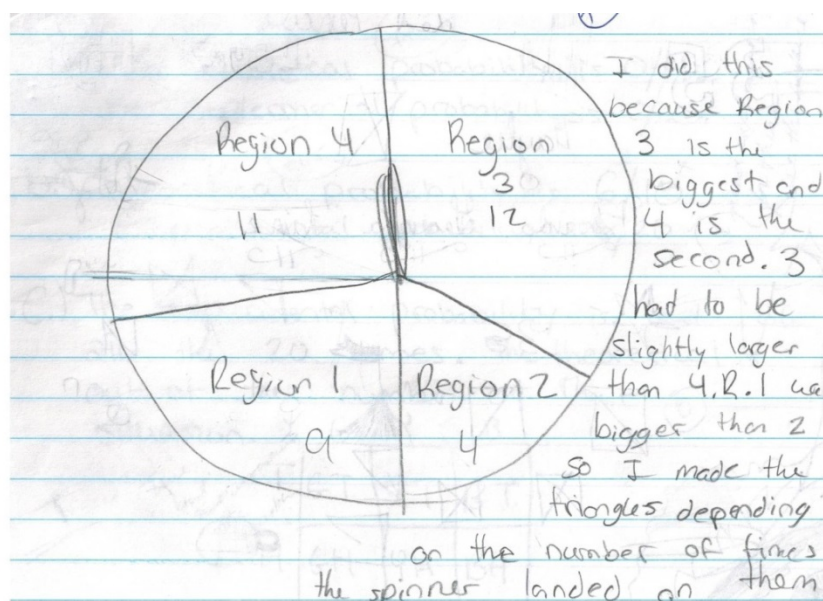


Figure 8. Inez's Response to Question 3

Ute and Henriët's capacity to notice the salient features of the spinner task are consistent with their comments regarding their attention to the details regarding Estelle's tree diagram. In both instances, these two students were able to sort the mathematical features provided in the task and choose those that offered an opportunity to bring meaning to the solution. I can't explain the fact that Inez did not notice the features of the written task, but fluidly discussed the features of Estelle's tree diagram. It is possible that she considered the values provided to be approximations due to the fact that they were experimental data. Or, perhaps she noticed the

proportional relationship among the values, but elected to assume a more general view of the information.

4.1.1.5 Positioning the students as authors

Positioning students as stakeholders by publicly identifying them with the claims, approaches, explanations, designs and other responses to problems is another element of authority as it is defined by Engle & Conant (2002). Many instances of this element of the principle of authority were present in the lesson transcripts. In fact, there were instances of the teacher positioning students as authors in each of the lessons included in this study. Common to most examples of the teacher positioning students as authors was the role the teacher assumed within the class. The teacher chooses not to evaluate student thinking and positions herself as a thinking partner. Students represent their own ideas and peers are expected to evaluate them; an established norm within the class. The participation pattern reflects atypical roles and responsibilities for both the teacher and the students. Several examples follow from a discussion during the first day of instruction related to probability. Near the conclusion of the lesson, the students were asked to graph the percent of heads versus the number of tosses, following a coin-flipping task. Students were encouraged to theorize regarding the shape of the graph when the number of tosses increased to a very large number. Students worked in small groups on the task for several minutes, then students were selected to present their graphs. Following Ed's presentation, the teacher makes the following comment.

Teacher: Not making sense to Dennis. This is Ed's reasoning. I didn't say that I shared his reasoning or anything about it. This is about what HE thinks so it is your job to ask questions if you disagree or don't understand, go up there. (day 1)

In this comment, the teacher attributes the thinking to Ed and clearly states that she will not be the person to evaluate his thinking. She assumes the role of neither the creator of the idea

nor the evaluator; atypical of traditional classroom. Similarly, in the following example, the teacher has monitored student work, and asked Bob to share his graph. She encourages Bob to talk about his creation; a move that offers Bob agency.

Teacher: Bob, you wrote the red graph. The red graph is yours. You want to talk about it?

These two examples represent an established norm within the class. That is, if the work or idea belonged to a student and it was selected for discussion, the student or group of students were encouraged to explain their thinking. Student work during this unit was discussed via a document projector, using large sheets of white paper, and on the white board. Making thinking public was an essential part of the class, and students eagerly presented their ideas. In addition to the teacher positioning students as authors, peers placed each other in the position to author ideas. As students exchanged ideas in both small and large groups, they authored ideas and critiqued the reasoning of each other. Examples of this behavior are discussed in more detail in section 4.2 wherein the principle of accountability is discussed.

The results of the student questionnaire support the results related to the ways the students and teachers enact the principle of authority. Students' views of their own opportunities for sharing authority were captured via question numbers 2, 4, 6, 8, 9, and 10 (Appendix A). It is important for the reader to recognize that some questions may provide information related to more than one of the principles, because of their integral nature.

Results of each question follow:

- 100% of students either agree or strongly agree on question 2,6,8,9
- 95% of students either agree or strongly agree on question 4
- 84% of students either strongly disagree or disagree

These results indicate the extent to which students were aware of their own authority within the classroom. Overwhelmingly, these results indicate that students were cognizant of their own capacity to assume authority in this environment.

4.1.2 Ways the teacher and students enact the principle of accountability

Research points to evidence of teacher moves that encourage students to be accountable to the teacher and other members of the learning community, through the implementation of classroom norms (Yackel & Cobb, 1996). The principle of accountability, being accountable to others and to disciplinary norms, implies that the teacher and other members of the learning community foster students' responsibilities to consult others in constructing understanding in a domain; it doesn't require acceptance of others' views, but responsiveness to them. "This principle is an expression of the value that each member of a learning community is not an authority unto himself, but one intellectual stakeholder among many in the classroom and beyond" (Engle & Conant, 2002, p. 405). Students who take their peers' ideas into account may be better positioned to persuade others of their own ideas, thus motivating further participation. In addition, being held to disciplinary norms helps to balance student authority and reduce the chance of students constructing haphazard responses to problems without peer review (Cobb & Hodge, 2002). Balance between authority and accountability is central to the principles of productive disciplinary engagement (Engle & Conant, 2002). That is, certain ways of communicating can in themselves affect the power among people.

In order to understand the way that accountability was enacted in the classroom, verbatim transcripts were analyzed. As I coded for instances of accountability, patterns began to emerge. That is, the instances had common features that allowed them to be grouped into categories; used

as sub-headings in this section. Among the patterns that emerged included: 1) the teacher placing student in a position to publically revise their thinking, and 2) students demonstrating intellectual courage to hold peers accountable.

4.1.2.1 Placing students in a position to publically revise their thinking

Among the student responsibilities that were apparent in transcripts of classroom lessons included the students making their thinking public. As a matter of routine, students presented their work and thinking to the class. Thinking was both a private and a public event; an individual and a community responsibility. The teacher's role then included positioning students to both make their thinking public as well as to revise their thinking as ideas changed. Implementing public thinking changes the authority for determining valid knowledge, from the teacher, to the student and community. In a classroom where authority is shared with students, multiple students may make their thinking public; listening to themselves as they talk. Engle & Conant (2002) posit that making sense of these ideas relative to other people's ideas encourages learners to consider how their ideas do or do not make sense in light of each other; prompting learners to revise their ideas for the better. An example of thinking being both a private and public event for students is shown in the following segment as students considered the Marble task (Figure 9) on day 5 and day 6 of the study.

Mrs. Rhee's math class was studying statistics. She brought in three bags containing red and blue marbles. The three bags were labeled as shown below.

Bag X: 75 red, 25 blue Bag Y: 40 red, 20 blue Bag Z: 100 red, 25 blue

Mrs. Rhee shook each bag. She asked the class, "If you close your eyes, reach into the bag, and remove 1 marble, which bag would give you the best chance of picking a blue marble?"

Which bag would you choose? _____

Explain why this bag gives you the best chance of picking a blue marble. You may use the diagrams above in your explanation.

Figure 9. Marble task adapted from the Quasar study (day 5,6) (Silver, Smith, & Nelson, 1995)

In this example, during exploration time when students were working in small groups, the teacher requested an explanation of a student and her partner. In response, Sydney (the student) offers a succinct explanation of the pairs' thinking.

Teacher: Ok. So tell me about this.

Sydney: (while she's writing...)The first method we had was we took 25 blue versus the total. And we took that and made it a percent cuz a percent is easier for us to compare. So we found that bag y was a better chance of picking out a blue. ..so that was our first method. Then our second method Lyla came up with this one (Lyla making faces to teacher). We simplified blue over the red and simplified and found out bag y. That bag y has the best chance.

This explanation, and others like it, demands that classmates consider their words carefully. Sydney, both explains the thinking while concurrently giving credit to Lyla for her solution method. This explanation served as a sort of rehearsal for a whole group discussion of problem solutions that took place the following day, (day 6).

During the day 6 class, several pairs were chosen to present to the class. Among the groups that presented were Ed and Estelle. Following Henriët's description of her method that included using a part to whole relationship, just as Sydney and Lyla had explained the day earlier to the teacher, Ed explains his method.

(2:54) Teacher: K. Ed. Would you and Estelle talk about this? First show your picture and then your graph. (Ed/Estelle go to the doc projector.)

Ed: So we picked bag y because it had the best portion of blue marbles to red marbles. Bag x had one blue marble to 3 red marbles; bag y was one blue marble to every 2 red marbles; and bag z was one blue marble to every 5 red marbles.

Teacher: (to class) So what is different about this than the one before? (referring to the last presentation) Talk to him.

Sydney: Basically, what you're doing . It's kinda like how Henriette did it. It's like one –third..oh, no it's not...I'm sorry...

Teacher: It's different.

Sydney: Ya, it's different. Sorry, because it's basically a ratio. You're taking one to three. So, its like one for every three marbles...what they did, they divided into a fraction.

Bob: They used the red and blue instead of the blue and total. (inaudible)

In this exchange, Sydney is thinking along with each of the presenters, but doesn't quickly notice the difference between Henriette's presentation of the part-whole relationship and Ed's presentation of the part-part relationship. She quickly corrects herself as she sees her own confusion; making sense of her own ideas in light of her peers' explanations. Bob summarizes the difference in methods for her following her expression of momentary confusion.

Students revising their own thinking was apparent during a partner quiz, administered on day 4, as well. One question, shown in Figure 4. 9, in particular caused students to consider their responses carefully.

“Juanita is holding five coins with a total value of 27 cents. A) What is the probability that three of the coins are pennies? Explain your answer. B) What is the probability that one of the coins is a quarter? Explain your answer.” (Lappan et.al, 2014).

Figure 10. Assessment item from the Partner quiz (day 4)

In the following sequence Inez and Henriët complete their quiz and had submitted it to the teacher, who glanced at the paper. She used the phrase “talk to me” as a way to hold them accountTable 4.8or explaining their response. As students began to talk, they determined that it was impossible to have a quarter as a part of the five coins: proving through exclusion.

(32:14) Inez: We’re not quite sure about the last one.

Teacher: So, so talk to me.

Henriët: So (inaudible) cause one penny is worth one cent? Or does it mean how many times 3 goes into 27?

Teacher: (reading the question) It says, ... “what is the probability that three of the coins are pennies? In other words can three of the ... So how do you make, with 5 coins, make 27 cents with 5 coins.

Inez: 3 pennies.

Teacher: That’s 3 cents.

Inez: You need....

Henriët: A dime...

Teacher: You need 24 cents.

Inez: Wait.

Henriët: It’s not possible...because you can’t have a quarter.

Teacher: So if it’s not possible, how do you record that as a probability?

Inez: Zero?

Henriët: Zero?

Teacher: Zero. If it’s impossible, there is a zero percent chance of happening.

The teacher becomes a thinking partner in this episode; reading and rewording the question, reminding them that three pennies is three cents. It is unclear whether Henriët was considering the need to have twenty-four cents, or if the teacher spoke too quickly and led her to

the need for twenty four cents. It seems that the act of encouraging them to talk about their thinking was enough to give them access to the solution. Henriët concludes that it isn't possible, revising her earlier solution. The teacher then prompts them to think about the way to record an impossible probability and confirms their conjecture that it is zero.

Similarly during the same partner quiz with Marcy and Estelle on the identical question a prompt to "just talk to me" resulted in the students thinking more carefully about their mathematical reasoning and revising their response.

(34:00) Marcy and Estelle (walk up to teacher and hand her the test) We're done.

Teacher: I'll take it. (she looks at it briefly) Now hold on a second. So show me how you're going to get three pennies here. You're going to use three pennies. Just talk to me for a second. How are you going to use three pennies to get 27cents?

Marcy: OOOH

Teacher: What will you have? What will the other coins be?
Three pennies....

Estelle: It'd be.....that wouldn't work.

Marcy : We did this with...we did that wrong. Thank you. (turned and returned to their seats with the paper)

In this short sequence, the suggestion for mathematical justification resulted in both students rethinking their response. Neither student finished a sentence before revising their thinking and correcting their response. Making thinking public and students revising their thinking is addressed further in section 4.2.3.

4.1.2.2 Students demonstrate intellectual courage to hold peers accountable

Similarly, in the topically related segment that follows in Table 4.4, taken from lesson ten, there was an extended discussion that included several significant features that are consistent with the

students having a high degree of both accountability and authority; represented by the upper right quadrant in Figure 4.10. (The task was presented near the end of day 9, but the majority of discussion was held on day 10.) Due to the extended nature of this discussion and my intention to illustrate the function of each utterance, I have included a line-by-line commentary that is intended to draw the reader's attention to the ways that the teacher and students enact the principle of accountability. By the end of this segment, it is apparent that several students have reconsidered their conjectures and have altered their conclusions as a result of listening to peer reasoning. Although I present this example as an exemplar of peers holding each other accountable, there are numerous examples of peer accountability throughout the data including the topically related segment already discussed in Table 4.2.

The task in which students are engaged is one that includes the investigation of an area model that represents the contents of two buckets of marbles of various colors, shown in Figure 4.10. The task challenges students to make sense of the provided area model and to notice the relationship between proportional reasoning and the area model. Of course, all three options provided in the task, could represent the contents of the buckets. The commentary to the right of each utterance (Table C) represents my view of the function of each comment through the lens of the principles of *accountability* and *authority*. Students repeatedly demonstrated intellectual courage and held peers accountable for their reasoning. In this whole group discussion, Dennis and Ed, partners in the task, are presenting at the document projector. The teacher asked Dennis to do the presentation, but he refused. When he refused, she asked his partner, Ed, to join him at the document projector for support. Dennis sat with his arms crossed, slumped in his chair, determined not to participate in the discussion. Table 4.4 represents the discussion that followed, with Ed doing the talking for the pair. Although the teacher tried to

hold Dennis accountable for engaging in the discussion, he provides an example of a student she could not engage in discussions most of the time. Dennis was not among the special needs students in the class. He was capable of participating but chose not to do so.

The area model below represents a different situation from Questions A and B. In this area model, $P(RY)=1/10$, $P(RB)=1/10$, $P(GY)=4/10$, and $P(GB)=4/10$. Use the area model and these probabilities to answer the following questions:

	Y	B
R		

4. Which of the following could be the contents of the two buckets? Explain your reasoning.

- a. 2 red and 8 green in bucket 1; 5 yellow and 5 blue in bucket 2
- b. 2 red and 8 green in bucket 1; 10 yellow and 10 blue in bucket 2
- c. 1 red and 4 green in bucket 1; 3 yellow and 3 blue in bucket 2

Figure 11. Two Buckets Task (day 9 and 10)

Table 7. Classroom Discourse Related to the Modeling Two Marbles Task

	Teacher	Student	Commentary
222	(22:00) Teacher: Dennis, come on, let's do it. Hey, there's no harm. If you need help from your group you need help. That's fine. Talk about what you know. Maybe somebody would like to come up with him. Maybe you'd feel a little more comfortable if you had a buddy to support and help answer questions. I know I am making you uncomfortable. That's all right. We are not going to let you hang. (Dennis sits in the chair, crosses his arms, and slumps. Ed joins him.) There is another chair there. Okay, so the two of you can do it. Dennis, which did your group vote for?		
247		Dennis- A	
248	Teacher: So you voted for two reds and eight greens. Let's just start with that. How does that represent two reds and eight greens? Do you have any idea about that?		The teacher tries to encourage Dennis to speak.
255		Dennis: No	

Table 7 (continued)

256		Ed: It's equal on both sides. (Writing under the document projector) $2+8=10$; $5+5=10$	Ed addresses the question.
	Teacher: So where are those numbers? On the pictures?		The teacher tries to help Ed make his thinking accessible to the class.
263		Ed: So this is bucket 1.	
	Teacher: Wait. We can't see what you're writing on. Let's move down here (repositions his paper under the document projector).		
271		Ed: And this is bucket 2. There are 10 on each side.	
	Teacher: Are there any questions? Don't ask me, ask him.		Students have hands raised. She tries to deflect the questions to Ed.
277		Henriet: The bucket one would be the red and the blue. Just that side.	.
280	Teacher: Henriet is saying this side represents bucket one and this side represents bucket two (pointing to the document projector image). What are you saying Ed and Dennis?		The teacher revoices Henriet then positions Ed and Dennis to answer Henriet's question.
		Ed: I'm saying all the stuff on this side is bucket one and this side is bucket two.	
292	Teacher: How about you Dennis? Do you see what he is saying?		She is checking for understanding the explanation of his peer.
		Dennis – No. It doesn't make sense.	He is expressing uncertainty about the explanation that has been presented by Ed.

Table 7 (continued)

297	<p>Teacher: Not making sense to Dennis. This is Ed's reasoning. I didn't say that I shared his reasoning or anything about it. This is</p> <p>about what HE thinks so it is your job to ask questions if you disagree or don't understand, go up there.</p>		<p>She is sharing authority with Ed; giving credit to his thinking process without evaluating it. She encourages him to author his own ideas. She invites peers to critique</p> <p>his reasoning and infers that the answer was arrived at via a process of reasoning that makes sense to Ed.</p>
308		Nancy: I agree.	She assumes authority, evaluates, and agrees with Ed.
		(many students begin to talk at once.) (25:41)	Students demonstrate their authority and boldly question the mathematical reasoning of the presenter.
	<p>Teacher – Yoohoo! if you have questions or you disagree, your comments need to go up here. Come on, let's go. There is a lot of uprising. What is bothering you?</p>		She is encouraging discussion and in doing so demonstrates to students that their ideas are of value and that knowledge is to be constructed together. Further she is encouraging students to use each others' ideas as resources and sends the message that thinking is a public, collaborative activity.
316		Henriet– I agree that it is A. But I don't agree with how you found it. (speaking to Ed) The whole thing is out of 10 I don't agree with adding 2+8 and 5+5, even though together you get 10. I don't really see that adding those... gets you your answer.	After considering Ed's mathematical explanation, she critiques his reasoning and in so doing, assumes authority and holds Ed accountable to restate his position or modify it.
		Nancy: I know. If you add the 2 and the 8 and got 10 and it equals 100 percent.	She also critiques his reasoning and in so doing assumes authority and holds Ed accountable.

Table 7 (continued)

330		Sydney - can I come up there and explain it? (She walks to the board and begins to explain to the class). (27:33) So what I did is $\frac{2}{10}$ would be in here and $\frac{8}{10}$ down here and $\frac{5}{10}$ on this one and $\frac{5}{10}$ here. You add and this would be $\frac{10}{10}$ over here and over here $5 + 5$ and this equals $\frac{10}{10}$ also. So, that is how I kind of think of it. But how we solved this is we multiplied it because if you multiply 10×10 you get 100.	She assumes authority and anticipates accountability via an unsolicited explanation. She uses the white board at the front of the class as a resource for her explanation .
346		Bob– I agree. All I want to do (writes on board). I think that might clear up a little bit of the confusion. (He writes the bucket 2 on the side of the area model opposite bucket 1.)	He assumes authority and shares a different representation that he believes is better than Sydney's.
353	Teacher – Now I have a problem with that because we don't have an area model anymore if you put yellow and blue down the other side. You have two separate things and one has to be on one axis and the other has to be on the other axis.		Noticing that there is a mathematical representation error, she assumes authority for a moment to correct the mathematical representation that will lead students to an incorrect conclusion. In this move, she holds students accountable to the discipline.
363		Ophelia : But if you are looking at those buckets, red, green and yellow would be in one bucket and blue would be in another.	She is clarifying her thinking.

Table 7 (continued)

368	Teacher – No. You have to look at bucket one as down the left side and bucket two across. So, what you need to notice is that across here, these are two equal parts. This part, and this part are equal. So you have the same number of yellow as blue. That is the take away here on yellow and blue. Yellow and blue have to be across the top and you can see that they are equal. So, for yellow and blue, I buy this 2 to 8 or $\frac{2}{10}$ to $\frac{8}{10}$. For yellow and blue, you have $\frac{5}{10}$ and $\frac{5}{10}$. Think about it like those clear plastic sheets where one is on top of the other so although this represents a proportion this way it can be a different proportion going this way and this way. You have two overlapping things.		She again assumes authority for a moment to correct a mathematical representation that will lead students to an incorrect conclusion. In this move, she holds students accountable to the discipline.
397		Sydney –(who is still standing at the board) So then after that, we just kind of did in our heads because it was really simple we just multiplied it to get 100 over 100.	She resumes authority and continues her explanation to the class.
404	Teacher – What fractions did you see? Those fractions?		Exploring mathematical meanings and/or relationships, she is holding Sydney accountable to the discipline.
407		Sydney – Yeah, $\frac{2}{10} + \frac{8}{10}$ you get $\frac{10}{10}$. $\frac{5}{10} + \frac{5}{10}$ get $\frac{10}{10}$ and then you multiply them together and you get 100 over 100.	She is responding to the request for accountability.

Table 7 (continued)

	Teacher – How does that help you decide which bucket?		The teacher is probing for mathematical explanation. She is giving Sydney the authority to think about it in her own way, while holding her accountTable 4.8or the communication of sound mathematical reasoning.
		Sydney – Well because they are both equal.	She is responding to the request for accountability.
	Teacher – What is equal?		She continues to probe in an effort to get Sydney to communicate sound mathematical reasoning.
		Sydney – if you are looking at it this way, this bucket would be 10/10 and this bucket would be 10/10.	She is responding to the request for accountability.
512	Teacher – So this only works if you have an equal number of marbles, is that what you're saying?		In an effort to extend student thinking, she is probing to determine if students are noticing a relationship between the area model and proportional reasoning.
		Ed – yes.	Demonstrates authority by evaluating the statement and agreeing.
	Teacher – So that is what you are saying? For this area model to work you have to have an equal number of red and green marbles and yellow and blue marbles. The two buckets have to be equal numbers. Does everyone in the class agree with that? No? Only one person doesn't agree? Henriet is standing her ground. Nya, let's hear you say something.		In an effort to generate discussion and continue to probe student understanding, she continues to try to encourage students to notice that the area model represents several proportional relationships.

Table 7 (continued)

533		Nya (walks up to the board) (31:24) I was thinking of all of this as a visual. I was thinking about to red and eight green and you look at it like this and the number is 10 so then you would have it like this. And then for the two red you would put it like, I don't know how to word it for it to make sense...(jumping up and down...making a circular motion with her hands)	She demonstrates authority through her attempt to provide an explanation of her own view of the area model resource. Her body is very animated but she demonstrates an incomplete explanation.
		Madia – Just draw it.	Expresses impatience with Nya.
		Nya – So what I'm trying to say is, I am thinking of it like a picture because if you look at this...and I know.	She clearly wants to make sense of the resource but can't verbalize an explanation.
552	Teacher – Let's think about this as an area model. So if this is a playground and this is the grass we are cutting. If this is $\frac{2}{10}$ what percentage of the grass are we cutting? And this? And this? It is 50 percent so then this is also 50 percent. It is 20% when you think of bucket one but it is 50% when you think of bucket two. You ignore it when you are thinking about the second bucket. Because the second bucket goes this way and it has two halves. So think about it in percent. So how does that change your thinking at all.		She attempts to make a connection to a traditional area problem by simplifying the context. Finally, she probes for understanding
575		Henriet – I still agree that it is A, but I don't agree with any of the ways that have been presented.	She demonstrates authority and critiques the reasoning of the peers and teacher.
	Teacher – Okay, so let's hear yours. (33:32)		She relinquishes authority and encourages Henriet to provide a mathematical justification.

Table 7 (continued)

581		Henriet – (walks to the board) She said both of these are $5/10$, but they're not equal.	She assumes authority.
585	Teacher – They are not equal. Sydney, do you see why those are not equal? Up and down they are not equal but they are side to side. Go ahead....		Revoices Henriet, then probes the understanding of Sydney. With this statement she has controlled the pace of the information that Henriet is providing.
		Henriet – So the way that I solved it – so what I did was $1/10 + 1/10$ is $2/10$ and then $4/10 + 4/10$ is $8/10$. And then $1/10 + 4/10$ is $5/10$. (pointing to segments in the area model)	She assumes authority and positions herself as an expert.
		Students– oh...., that makes perfect sense!	Students evaluate her thinking and concur.
600		Bob – you can see perfectly then if you go across...	He evaluates her thinking and concurs.
		Students (several) – That makes sense.	They evaluate her thinking and concur.
		(students begin to clap)	Students demonstrate appreciation to Henriet for sharing her thinking process. They all assume authority by evaluating her thinking and simultaneously position Henriet as an expert.
	Teacher – That's better than I did, Henriet. Okay, so then the question is does it have to be...why does two and eight work? Why does 2 here and 8 here work (pointing to the area model) Just think of the reds and the greens. Does it work? Dennis said it works. Why does it work? Estelle, why does it work? Why does two reds and eight greens work here?		She positions Henriet as an expert and positions herself as teacher as partner, engaged in the same intellectual work. (Tabak/Baumgartner). She is probing to extend their thinking and in so doing, holding students accountable to herself and the discipline.

In addition to Dennis' lack of engagement, perhaps the most significant feature of the discussion is the ease with which other students respond to mathematical ideas presented by their peers. Henriet, Nancy, Sydney, Ophelia, Nya, Madia, and Bob make contributions to a discussion that began with uncertainty expressed by Dennis regarding Ed's explanation. During this discussion, which proceeds for more than ten minutes, several students in succession walk to the white board while the teacher and the rest of the class look on. They are both eager to present their ideas and seemingly unaffected by whether their conjectures are correct or incorrect. In addition, their responses reflect that they are listening to the entire conversation and engaging with the thinking of their peers. Their refutations of their peers' ideas is a significant indication regarding the way they believe truth is established in mathematics. They seem to agree that truth is not established by the teacher or a book determining whether the answer is right or wrong, but by providing evidence to support or disprove a conjecture. Throughout the segment, it is apparent that they had been listening to peers' and the teacher' attempt to explain the mathematical thinking. However, until Henriet explained her thinking, they could not make the connection between the area model and the proportions represented by the problem. Henriet maintains her intellectual courage through the course of the discussion. The repeated assertions of other students did not incline her to revise her thinking and in fact allowed her to explain and challenge her peers' mathematical logic. Henriet agrees with the answer provided by Ed, but not his method of solution. She says, "I agree that it is A. But I don't agree with how you found it." Henriet seems to be following Ed's explanation of his mathematical thinking. Later, Henriet explains her own thinking; resulting in student applause. The fact that students clap at the conclusion of Henriet's explanation indicates the extent to which students are engaged in the

discussion. The intensity of student engagement to which Engle and Conant (2002) refer is apparent in this segment.

It is interesting that despite the enthusiasm of the class, and the number of speakers in this segment, that Dennis had not engaged publically.

Throughout the discussion authority and accountability seemed to be reflexively related. Without the distributed authority, students could not hold each other accountable. Each student exhibited authority as he critiqued the reasoning of peers. However, without the accountability, a discussion of this magnitude and detail may not have occurred. Students held each other accountable for clear explanations that reflected the area model.

Several moves by the teacher encouraged students to persevere in problem solving while holding them accountable to the community, knowledge, and rigorous thinking. First, she uses Accountable Talk moves (O'Connor & Michaels, 1993), (Appendix F) to probe student understanding and generate discussion. She says, "The two buckets have to be equal numbers. Does everyone in the class agree with that?" Later in the segment she says, "So this only works if you have an equal number of marbles. Is that what you're saying?" Both of these examples serve to generate discussion and encouraged students to examine their own assumptions. Second, she assumes total authority when Bob presents a model that she believes will serve to undermine the direction of the discussion to that point. She makes an in-the-moment decision to keep the class from following Bob's thinking; refuting his assertion and redirecting the discussion back to points discussed by Sydney. In that move, she offers to distribute authority, but quickly reassumes it in an attempt to keep the mathematical discussion progressing toward her goal of having students notice the connection between the area model and proportional reasoning.

In the following comment wherein she uses the word “notice”, she draws attention to the size of the area model portions. She says, “So what you need to notice is that across here, these are two equal parts.” She has both assumed authority and encouraged students to pay attention to the details of the construction of the area model. A few sentences later, she reminds students of a previous day where plastic transparency sheets had been used by students to represent the contents of each bucket. Using that resource, she provides students with structure that may be of use in making sense of the problem at hand.

The student questionnaire supports the results reported related to the principle of accountability. Question numbers 3,4, and 6 address accountability as well as several other principles.

How much do you agree with the following statements about your teacher in your Math class: My teacher: (circle the answer that reflects your opinion)

3. Often requires me to explain my answers.

Strongly disagree Disagree Agree Strongly agree

4. Encourages us to consider different solutions or points of view.

Strongly disagree Disagree Agree Strongly agree

6. Expects us to work together to solve problems.

Strongly disagree Disagree Agree Strongly agree

The results of student answers to these questions are as follows:

- 95% of students agree or strongly agree to question 3 and 4
- 100% of students agree or strongly agree to question 6.

The results indicate that students are aware of the expectation that they work together and consider others’ point of view as a peer explains their thinking. The students not only understood the expectation, but also were able to enact the capacity to critique the reasoning of others.

4.1.3 Ways the teacher and student enact the principle of problematizing

Engle & Conant (2002) discuss the importance of “problematizing” as the third core idea in their framework. Engle (2011) describes problematizing as, “any individual or collective action that encourages disciplinary uncertainties to be taken up by students” (p. 6). She further describes problematizing to include the extent to which genuine uncertainty is engendered in students, that problems are not easily resolved, that problems embody “big ideas of the discipline”, and that they are related to a topic that is of some interest to the learner. In order to succeed in problematizing, a teacher must create an environment where students must persevere together toward a common goal. Discourse among students is truly necessary in an environment that embodies the principle of problematizing because a course toward solution is not apparent. Students genuinely need to talk in order to determine a solution path, draw a conclusion, or synthesize their work. Problematizing describes a purposeful choice by the teacher in terms of the kinds of tasks students will engage and the way the tasks will be enacted. In other words, problematizing includes choosing tasks that encourage students to both interpret them and persevere in solving them, using available knowledge and resources. Genuine uncertainty must be created within students to have enacted the principle of problematizing. Congruent with other research that draws a connection between discursive participation, the related teacher practices that influence student learning, and the mathematical task selected by the teacher, problematizing is a central theme (Leinhardt & Steele, 2005; Stein, Smith, Henningsen, Silver, 2000; Silver, Smith, & Nelson, 1995; Smith, 2000; Lotan, 2003; Hiebert & Wearne, 1993; Kieran, Forman, & Sfard, 2003).

4.1.3.1 The task

Research collectively points to the importance of the task in creating a sense of uncertainty in students. A mathematical task is defined as a set of problems or single complex problem that focuses students' attention on a particular mathematical idea (Stein, Grover, & Henningsen, 1996). It is the task that provides something worthy of talk while promoting or discouraging students to explore deeply the intended mathematical goal. Although task selection and problematizing are not synonymous, problematizing largely depends on task selection and the enactment of the task by the teacher. This idea will be further explained in subsequent paragraphs.

Each of the tasks used in the study are illustrated in Appendix I. Table 4.5 represents a visual description of the task, the way it was used with students, along with the cognitive demand as described by the Math Task Analysis Guide (Stein, Smith, Henningsen, & Silver, 2000), shown in Appendix C.

Table 8. Summary of Tasks Utilized in this Study

Data Collection Day	Task by name or number	Cognitive Demand³	Setting utilized
Collection day	Task	Cognitive Demand	Setting utilized
1	Problem 1.1	PWC	In class
	Orally assigned graphing task	DM	In class
2	Problem 1.2	PWC	In class
	Develop definition	DM	In class
	Homework problem 1	PWOC	Homework
	Homework problem 2	PWOC	Homework
	Homework problem 3	PWC	Homework
	Homework problem 4	PWC	homework and in class
	Homework problem 5	PWC	Homework
3	Problem 6A	PWC	In class
	Problem 1.3	PWC	In class
4	Partner Quiz		Assessment
	Problem 1	DM	In class
	Problem 2	DM	In class
	Problem 3	DM	In class
	#19 a,b,c,d	PWOC	Homework
5	Marbles task (Quasar)	DM	In class
6	No new tasks		
7	Problem 2.3	DM	In class
	Cafeteria problem	PWC	In class
8	Marbles task (NAEP)	DM	Assessment
	Sticky Gum Problem	DM	In class
9	Problem 4.1 A,C	PWC	In class
10	Problem 1-4	PWC	In class
11	Yellow face cube task	DM	Assessment
12	Making Purple	DM	In class
13	Problem 4.3	PWC	In class
14	Problem 15- One and One simulation	DM	In class
15	Assessment – question 1,2,3	PWC,PWC,PWC	Assessment
	Caves Paths	PWC	Homework and in class

³ DM=Doing Mathematics, PW=Procedures with Connections, PWC= Procedures w/o Connections (Stein,Smith, Henningsen, & Silver, 2000)

The results indicate that a high percentage of tasks utilized in this unit of study fall within the categories designated as high cognitive demand: either procedures with connections or doing mathematics. Specifically 9% of tasks were of low cognitive demand, and 91 % of tasks were classified as tasks of high cognitive demand. Within those tasks of high cognitive demand, 52% were considered as procedures with connections and 39% were classified as doing mathematics. All three tasks having low cognitive demand were assigned for homework as procedural practice. Several of the tasks that were classified as doing mathematics were from the Connected Mathematics curriculum (Lappan et.al, 2014), while others were integrated from released NAEP items or items from the QUASAR study (Silver, Smith & Nelson, 1995).

4.1.3.2 Student uncertainty

The authors of the Math Task Analysis Guide indicate that the solution process is unpredictable in a task of that qualifies as doing mathematics, and that students may experience “anxiety” as a result. Engle and Conant (2002) use the word “uncertainty” to refer to the same student disequilibrium. Once students understand the task, they may still not be able to arrive at a solution path quickly or know what to conclude, or how to justify their reasoning. The struggle requires perseverance on the part of students and often includes uncertainty; what Engle & Conant (2002) refer to as uncertainty as to what to do or uncertainty as to what to conclude and what the CCSS-M, Mathematical Practices refer to as persevering in problem solving. All of these sources agree that this struggle is productive and necessary in the process of students’ construction of mathematical meaning. In this study, uncertainty was very common, and present during each day of instruction.

In regards to the results of this study, I coded all verbatim transcriptions for student uncertainty. Specifically, I coded instances wherein students orally expressed uncertainty such

as, “ I don’t understand” or “I don’t get it”. Every class transcript had expressions of student uncertainty. It is likely that student uncertainty resulted from their engagement with tasks of high cognitive demand that were utilized in every class. In the coding process, I did not distinguish between uncertainty as to what to conclude, how to justify what one is doing, what to do, or a combination. An example of the way student uncertainty was apparent follows.

This struggle and uncertainty as to what to conclude is apparent in the excerpt of the third class during the review of homework. The task (homework problem 4) that students addressed was, “Len tosses a coin three times. The coin shows heads every time. What are the chances the coin shows tails on the next toss? Explain.” (Lappan et.al, 2014). This task challenges students to consider events that are independent or dependent, an unfamiliar concept to students, and asks them to consider prior experience or knowledge that might be useful. The student (Henriet) who begins the discussion assumes authority and offers a conjecture. Her erroneous response provides a point for further class discussion and the airing of student uncertainty.

Henriet: Well I said it would be one-fourth, cuz I (teacher writing $\frac{1}{4}$ on board) I said if he had already had head three times....i would think that the fourth time it is more likely he’d get a tail. (teacher sitting with a group near the board) (Pause)

Lyla: I think it’s still like a 50% chance because even though it’s heads three times, that doesn’t really matter. Because no matter how many times you flip it, it’s going to be a 50-50 chance because it’s a fair game, it’s equal chances. (Teacher still sitting)

Bob: I have to agree with Lyla. I mean you’re using the same coin. The chance is always going to be 50-50. I mean, there’s a heads there’s a tails (Bob, crouches on his seat) . There’s only two chances. Two choices I mean. (day 3)

Lyla and Bob have disagreed with Henriët and have attempted to justify their conjectures using mathematical reasoning. They have necessarily elaborated their assumptions and have held Henriët accountable for her mathematical reasoning. The teacher has chosen to share her authority by allowing the students to provide comments to Henriët, and by sitting quietly with another student group. In the next exchange, the teacher generates further discussion by positioning students on two sides of the argument: Henriët's side or Lyla and Bob's side. Following a clarification of what constitutes a fair coin, both Madia and Ute express some uncertainty. Both are assuming authority by attempting to evaluate student contributions. Ute has clearly immersed himself in the thinking of both groups of students and summarizes each position.

(6:56) Nancy: I agree. But can't you like have a coin that has both on one side?

Teacher: We're assuming they're not biased.

Nancy: Ok . Unless it's like that. There's always going to be a 50-50 chance.

Teacher: Who agrees with Henriët? Anybody agree with Henriët? (Madia raises her hand)

Madia: I'm both. I can't decide.

Ute: I don't agree with it, but I can see where it's coming from because it's just common sense to think ok, well, if I'm going to flip it four times that ..and I already had three heads that y'know there's probably going to be a tails because they ARE equal. But at the same time, like Bob said, it's the same coin. The same deal. It's always going to be 50-50 so it really could be either way. So I see where she's coming from but I don't necessarily agree with it.

Madia: I get both sides, but I'm not sure which one I agree with. (day 3)

After a further expression of uncertainty, the teacher returns to Henriette to probe her understanding, now that she has considered an alternate position. Although confident in her initial response to begin the discussion, Henriette now exhibits uncertainty. She is publically reconsidering her position.

Teacher: Henriette, what do you think, after hearing the other side?

Henriette: I thought about that and I see it from both sides...like ...I don't know.

The teacher then assumes authority to interject the words independent and dependent events. Once students realize that the fourth coin is not dependent on the results of the previous three flips, they arrive at a consensus position regarding a 50% probability of flipping a head on the fourth coin.

Teacher: (stands and walks to the white board) How about these words? How about this word? You have to ask yourself if these are (writing) INDEPENDENT or DEPENDENT events. What is ...I don't know how to spell this... What is independent versus dependent events? (sits back down with a group) Does the fourth one depend on how the other three went?

Henriette: No

Teacher: No, it doesn't depend on it, so it is an INDEPENDENT event. It's not dependent on the first three. So.... (pause) What probability does it have?

(8:58) Nadeem: 50%

Teacher: You think?

Bob: Ya

Teacher: Does everybody think it's fifty percent?

Nya, Kasey, Henriette, and others shake heads yes. (day 3)

This segment emphasizes the importance of the task to the principle of problematizing. The task presented caused students to be uncertain and to monitor their own cognitive processes, as was apparent in Ute's summary of the two sides. He is reflecting aloud about his perception of each student's explanation as he considers both sides carefully. The task has encouraged uncertainty which was presented by students confidently and was in fact embraced by fellow students. The student and teacher reaction to uncertainty was respectful and took the form of mathematical reasoning.

Similarly when the class undertook the following task on day 5, uncertainty was expressed and students and teacher responded with mathematical reasoning. The task presented was: "Mrs. Rhee's math class was studying statistics. She brought in three bags containing red and blue marbles. The 3 bags were labeled as shown. Mrs Rhee shook each bag. She asked the class, "If you close your eyes reach into a bag and remove 1 marble, which bag would give you the best chance of picking a blue marble? Notice that the bags don't have the same number of marbles in them. So you need an explanation and two representations." (adapted from Silver, Smith, & Nelson, 1995) (day 5)

As students undertook the completion of this task in their small groups, they used a variety of solution methods, as requested; demonstrating authority. Some determined the fraction of each bag that blue marbles, one group utilized percent of blue marbles, several groups used part-to-part ratio comparing blue to red marbles, while another scaled up each bag so that the same number of marbles was contained in each bag. Student solutions are shown in Appendix X. Groups generally had little problem utilizing one strategy, but were challenged to produce two solution methods.

When it came time for sharing solution methods, Ed was chosen among the presenters. He and his partner were the third and final pair to present. He illustrated the ratios of blue marbles to red marbles in each bag represented in the form of line graphs (Figure 4.11). Marcy was uncertain regarding Estelle and Ed's method. The uncertainty itself is an important element, but perhaps more importantly, as in the last segment, her uncertainty is met with mathematical reasoning by another student and the teacher who work together; each providing different representations, to help Marcy make sense of his method.

Marcy: I don't understand how you got that.

Teacher: She's not understanding how you got that. Can you talk more?

Ed: For bag y it's double. For the red ones there's two for every one (showing his graph under the document projector).

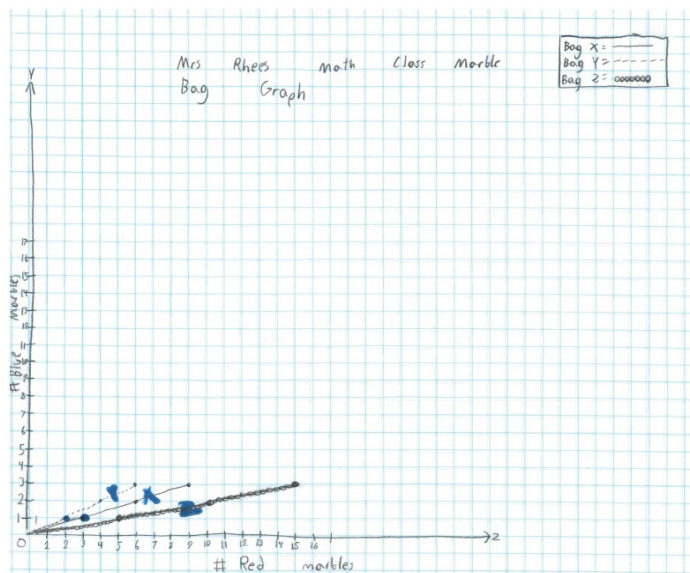


Figure 12. Estelle and Ed's Marble Graph (day 6)

As the exchange begins, the teacher uses Accountable Talk (O'Connor & Michaels, 1993) to encourage Ed to clarify his explanation. She revoices Marcy's expression of uncertainty, then asks Ed to say more so that Marcy can engage more profitably from his idea.

Teacher: Look up here. This might help you, Marcy. Twenty to forty, turns into a ratio of one to two (referring to the ratio that is one the white board) Which is what you're saying, right? (talking to Ed)

Ed: Exactly

Teacher: So how does that one to two, show up on that graph? How does it represent on the graph?

Ed: It goes over 2 and up one.

Teacher: Show us. Point to it with a pencil. (Ed is showing on the document projector) So every 2 it goes up 1.

In this exchange, the teacher and Ed share authority in providing Marcy with a response to her expression of uncertainty. The teacher addresses her uncertainty initially by offering a familiar representation, a ratio, to Lauren. She shares authority with Ed when she says, "Which is what you're saying, right?". Ed and the teacher become partners in helping Marcy to see the connection between the ratio and the graph when he answers, "Exactly". The teacher uses Accountable Talk to encourage Ed to say more regarding the tie between the proportional reasoning representation and the line graph. The teacher opens a conversation for Ed to explain the relationship between the proportion that was written on the whiteboard and the graphical representation that he is illustrating when she says, "So how does that one to two show up on the graph? How does it represent on the graph?" Although she recognizes that Ed understands the concept in question, she asks Ed the question for Lauren's benefit. He becomes the authority in this instance. Ed demonstrates his method of "goes over 2 and up 1" in an effort to help Marcy follow his mathematical reasoning. The teacher requests that Ed provide a very deliberate

explanation using his graph when she says, “Show us. Point to it with a pencil.” She is holding Ed accountable for an explanation that is complimented by a visual demonstration in the use of his graph. In addition, by using the word, “us” she has assumed the position of teacher as partner, implying that she, too, needs the explanation. By assuming that stance she sends the message to Marcy that her question is worthy and there will be others that benefit from further discussion. In the next line, the teacher begins the sentence with “so”, further positioning herself as teacher as partner (Tabak & Bumgartner, 2004). By using the discourse marker, “so” prior to revoicing (O’Connor & Michaels, 1993) Ed’s comment she not only rephrases his remark, but she does so in a way that attributes the “revoiced” statement to Ed and entitles him to negotiate her interpretation of his remarks. This use of teacher as partner and revoicing is what Tabak & Baumgartner (2004) call symmetry fostering participant structures. That is, setting up symmetry between the students and teacher regarding the respective authority over knowledge construction.

A key point to which I would like to draw attention is the entwined nature of the task, the expression of uncertainty by Lauren, and the way that authority and accountability were used in this segment. The task itself provided the opportunity for multiple solution methods. Student groups addressed this task using several representations. When Ed provided his graphical representation and explanation, Marcy expressed uncertainty; not following his mathematical thinking. The way the uncertainty was handled by Ed and the teacher are important pedagogical features that illuminate the bond between the principles of authority, accountability and problematizing. Without the attention to sharing authority, the teacher may have become the authority of the knowledge. Her talk may have been in the form of evaluation of Ed’s mathematical thinking or of Lauren’s lack of understanding. In this exchange, it was neither.

She has suggested to students, through sharing authority, that learning includes both mathematical analysis and extension of ideas.

The integral nature of the task to the principle of problematizing is apparent in the two segments discussed. Because a well-chosen task provides some degree of uncertainty, students necessarily are engaged in talking about their mathematical thinking. The strategic choice of representations for class discussion provides an opportunity to both air uncertainty and to make public the construction of meaning among representations. Students must persevere in problem solving in order to arrive at solutions that are accepted by their peers (CCSS-M, 2010).

Results of the student questionnaire point to the awareness that students developed regarding the types of tasks they were provided, and the expectation to persevere in problem solving. Question 5 asks students to circle a response ranging from strongly disagree, disagree, agree, or strongly agree related to the phrase, “Encourages students to stop working when the work gets hard.” Thirteen students voted strongly disagree, 4 voted disagree, 1 voted agree, and 1 voted strongly agree. Likewise, regarding the phrase, “Gives us work in class that is challenging”, the overwhelming majority recognized that tasks were not routine; 11 voted agree and 5 voted disagree (Not all students answered the question.) Offering tasks that were of low cognitive-demand, but simply too difficult for students might result in a similar student response. However, I contend that because I have demonstrated that the majority of tasks were of high cognitive demand; that was not the case. Question 8, “Wants us to become better thinkers, not just memorize things” also points to the selection of tasks. Twelve students strongly agreed and 7 agreed. Students apparently recognized that thinking was valued and that struggling to complete the task was acceptable.

4.1.4 Ways the teacher and the students enact the principle of resources

In the analysis of the preceding segment, I chose to background the resources used by the participants. I will highlight them in this section, to demonstrate that Engle & Conant's (2002) claim that problematizing is balanced by the principle of providing resources to students. With insufficient resources, students are unable to act and may be overwhelmed with the challenge; with too many resources provided, the problematic nature of the task is diminished. Engle & Conant (2002) describe the provision of resources as a necessary fourth element in the support of productive disciplinary engagement. They define resources very generally and include anything or anyone that may be seen as necessary to support the embodiment of the other principles. Resources may be as fundamental as providing students with time to solve meaningful problems (Henningesen & Stein, 1997) or may be more specific to the task. Engle and Conant (2002) cite examples of providing resources; the provision of home-based modes (talk that is consistent with the style at home) of discussion in support of problematizing content as well as the provision of models and norms in the classroom. Peers, physical manipulatives, teacher questions, and anything that might amplify a student's capacity to problem solve qualify as a resource using this broad definition.

In this study, physical manipulatives and graph paper were always available. Students left their seats to gather both when they felt they needed it. I often drew attention to it, but most often did not need to, as students used them without prompting. Peers were also utilized daily because most work was small group or whole group; little individual work time was provided. Some resources, like prior tasks or experiences on which students drew, I didn't predict or plan for; they became resources through connections within the student's mind. Table 4.6 represents the resources on which students depended.

The student questionnaire points to the students' awareness of the provision of resources. Question 11 requests a response ranging from strongly disagree, disagree, agree, or strongly agree regarding the phrase, "Makes resources (graph paper, spinners, books) available to us in case we need it. All nineteen students voted strongly agree or agree, indicating they recognized that physical tools were offered as a matter of routine.

In the segment described in Table 4.1, wherein Ed provided a representation that caused Marcy to express uncertainty, resources were critical to the discussion (Figure 4.11). Ed and the teacher both serve as ideational (Nasir & Cooks, 2009) resources for Lauren; both offering their ideas about the ways the representations were related. The teacher attempted to help Marcy move from a representation she understood, ratios, to something less familiar, the graph. The ideas she presented were resources as well as the representation, a ratio, presented on the white board. Likewise, Ed provided both a visual resource, his line graph, as well as some idea of the way the mathematics made sense to him. In the end, however, it was Lauren's responsibility to make sense of it all. It was necessary for her to persevere, assume authority and responsibility for her own learning. In fact, it was that authority or intellectual courage that caused her to express uncertainty at all. Her question, was in fact, an expression that implied that she was trying to follow his mathematical thinking, but couldn't. At the foundation of the entire lesson was the task. By challenging students to engage in a task that offered a number of solutions, and enacting it in a way that encouraged students to make connections among the strategies, students needed to grapple with both their solution method and the mathematical thinking of their peers. The task created uncertainty regarding the connection among the representations; uncertainty that may have been overwhelming without the resources provided by the teacher and Ed.

Table 9. Visible Resources Utilized by Students During Class Work

Data Collection day number	Task by name or number (Cognitive demand)	Resources utilized by students
1	Problem 1.1 (PWC)	Coins, peers, teacher, table structure for record keeping
	Orally assigned graphing task (DM)	Graph paper, peers, teacher
2	Problem 1.2 (PWC)	Cups, calculators, peers, paper for record keeping, teacher
	Develop definition (DM)	Peers, Venn diagram on the white board
3	Problem 6A (PWC)	Peers, teacher
	Problem 1.3(PWC)	Coins, peers, teacher
4	Partner Quiz: Problems 1,2,3 (PWC)	Partners, NCTM Core Tools on the Smartboard,
5	Marbles Task (Quasar) (DM)	Graph paper, peers, prior proportional reasoning tasks, chips (for marbles)
6	No new tasks	
7	Problem 2.3 (PWC)	Coins, peers, teacher
	Cafeteria problem	Past experience with the tree diagram, peers, teacher
8	Marbles task (NAEP)	
	Sticky Gum Problem	Chips (for gumballs), peers, teacher
9	Problem 4.1 A,C	Experience with arrays, peers, teacher
10	Problems 1-4, page 80	Peers, teacher, task from day 9
11	Yellow-face cube task	Peers, teacher
12	Making Purple	Experience with arrays, peers, teacher, visual on white board
13	Problem 4.3	Experience with arrays, peers, teacher, connection to basketball experience for some.
14	One-and one simulation	Dice, spinners, area models
15	Assessment questions 1-3	None visible.

In order to strengthen the validity of the point I have just made, regarding the importance of the task selected as it contributes to problematizing, and the role of resources in balancing the challenge, the following segment represents another group of three students who are engaged in the same marbles task. In this segment the teacher had spoken to students as she circulated from

group to group. Nadeem demonstrated an incomplete understanding of the relationship between the probability of choosing a specific color and the proportions of each color. What follows is discussion between the other students in Nadeem's group and the teacher regarding formerly – experienced tasks that they envision as related to the one at hand. Through the discussion they offer their former experience as a resource from which Nadeem may draw.

Teacher: So Matt, Dennis, and Nancy, what are you doing? What are you thinking? I come in late all the time and I miss part of the story.

Nadeem: (dropping chips) We found the number of chips so we divided The blue by the red in order to find the percent of ...we found bag y is the most effective in pulling out a blue...if you double this you have more of a chance of hooking red..the same thing holds for this (overlapping)

Teacher: I'm a little bit confused though, because I thought when you had a 50% chance of something like the coin toss you'd need an equal number to give you a 50% chance. The 50% probability that you are saying is confusing me a little bit because I would think if you had a 50% probability there'd be 20 blue and 20 red. (day 5)

The teacher's first questions, position her as someone interested in students' thinking, not just their answer. Nadeem responds with a statement that causes the teacher to believe he is thinking about the chips he is using to represent marbles, in the same way as the students had considered their recent experience with coins. She is wondering if he notices the difference in the likelihood of choosing a blue marble and the likelihood of flipping a head with a fair coin. Her comment to Nadeem and his partners positions her as teacher-as-partner through the use of an expression of uncertainty. Her comment begs for an explanation from someone in the group; an explanation that might serve as a resource for Nadeem. Dennis and Nancy respond.

Dennis: There are 60 total and 20 of them are blue. So it's one third.....point 3333.

Nancy: That's what I was thinking.

Teacher: So I think what you were thinking of is taking a ratio.... a part to part relationship, not a part to whole relationship. Do you remember anything you did, you did like this before?

Dennis and Nancy make the point that the part-to-whole relationship shows one-third blue marbles. The teacher's response is erroneous in that she is thinking they are considering a part-to-part relationship, but her comment still points them all to consider a prior experience, related to proportions, that she thinks might be useful as a resource and is consistent in keeping with her goal of students understanding the relationship between proportions and probability.

Nancy: Like this year?

Teacher: Ya, this year.

Dennis: Was it the orange concentrate thing?

Teacher: Ya. The orange juice concentrate thing was kinda like this, right? You had water and orange juice and you had a ratio of the relationship and you also had a part to whole relationship, right? Ya, it's similar.

Dennis recognizes a relationship between the marbles task and the orange juice task that the class had completed months prior. The teacher positions Dennis as the expert regarding the relationship between the two tasks when she verifies his connection and expands his response to include the idea of a part-whole relationship as a commonality that she notices in both tasks.

The lesson that transpired during day 15 also included the use of resources albeit different. This lesson exemplifies the balance between using resources and problematizing as one considers the principles of productive disciplinary engagement. The class spent roughly twenty three minutes discussing one homework problem, #15 on page 85 of Connected Mathematics (Lappan et.al, 2014). The question refers to the area model from problem #14 that is shown below problem 15 in Figure 4.11. Resources were central to the engagement of students. Each of the "maps" shown in Figure 4.11 was posted on large white paper on the board

as students arrived along with a copy of the area map. Posting the maps on the board provided students with a common reference to center class conversation. Having a large model allowed students to explain their thinking via walking to the front of the room and pointing to features of each map, later in the class. The teacher asked students to vote for which area map they considered to be a “match” with the area model, early in the class. Using this resource, students were encouraged to take a position. The class then engaged in a discussion regarding different groups’ explanations. This position-driven discussion was made possible through the use of the resource that included encouraging students to align with one posted figure. Had the teacher just indicated the right answer to students, the discussion that followed would have been lost. The provision of the resource was key to actually producing uncertainty and the need to explain mathematical thinking.

These examples indicate ways that the teacher and students have enacted the principle of resources (Engle & Conant, 2002). The resources discussed in the previous paragraphs included the prior tasks in which students engaged, peers, the teacher, a variety of representations of mathematical thinking, and material resources such as chips, the white board, video simulation, and document projector. The provision of resources made the tasks doable for students and have aided in the connection among mathematical ideas.

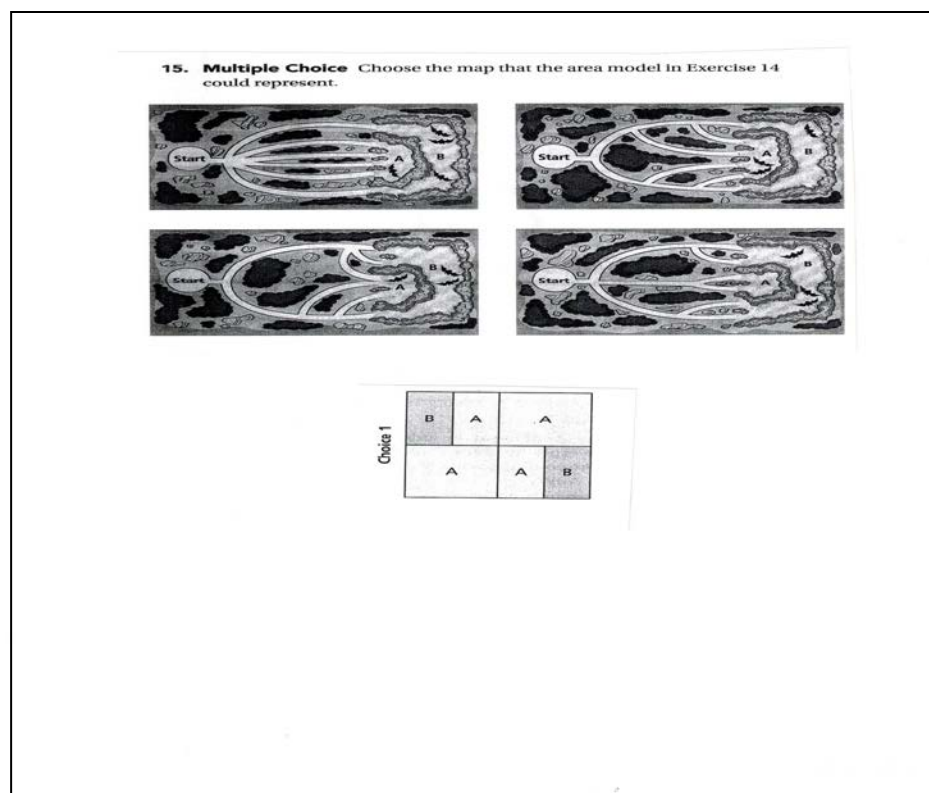


Figure 13. Day 15 Task (Task #15, p. 85, *Connected Mathematics, What do You Expect?* (Lappan, et.al. 2014))

Engle and Conant (2002) make clear the importance of implementing all four principles concurrently in order to accomplish *productive disciplinary engagement*. In the preceding results sections I have provided exemplars drawn from every lesson demonstrating each principle individually, in an effort to explicate ways that the principle was evident in the instructional environment. I highlighted some features while backgrounding others because it was important for readers to gain an understanding of the ways the teacher and students enacted each principle. Section 4.1.1, the section devoted to the results related to authority, and section 4.1.2, the section that addressed the principle of accountability, both utilized a lesson segment from day 7 described in Table 4.2. In those sections I described the ways that the teacher and students enacted the principles of authority and accountability in the same segment, but I only hinted at

examples of problematizing and the use of resources in the column designated as *function/commentary*. In the following paragraphs, I highlight the ways that problematizing and resources are also present in that same segment. The segment provides one example of the presence of all four principles synchronously. I argue that I could make this point for every lesson in the study.

As a reminder, in the day 7 lessons, Estelle was sharing her tree diagram with the class during the highlighted segment. The tree diagram that she designed had attracted the attention of the teacher as she monitored student progress in small groups. The tree diagram served as a resource that helped Estelle make her thinking public and afforded her classmates the opportunity to explore mathematical meaning as they investigated and discussed the possible meaning of the representation (lines 426 to lines 622). In addition, the students served as resources for each other in this segment. For example in lines 595 through 605, Lyla assumes authority and restates the logic behind each representation. In so doing, she offers the other students an idea to consider; a resource for further consideration. The tree diagram resource balanced the challenge associated with the task. As students engaged with the visual representation that the tree diagram provided, they were afforded access to the challenge of the task. The resource offered each student a thinking tool as well as a common public thinking tool as Estelle shared her tree diagram.

Finally, problematizing was prominent in this segment. The task itself presented students with the opportunity to air uncertainty and to make public the construction of meaning using the tree diagram. Students persevered in problem solving to arrive at a solution that was accepted by their peers, and were positioned as decision makers, resulting in the authentic need for classroom discourse. The task selection, combined with careful teacher monitoring of student thinking as

they worked in small groups, allowed for a robust discussion focused on the tree diagram that Estelle had produced.

As one considers the four principles enacted simultaneously, the reflexive relationships among the principles becomes apparent. Resources provided access for students in engaging in the task. Both the resource and the task itself offered students something worthy of a discussion. The discussion allowed students to hold one another accountable and enabled students to assume authority, traditionally held by the teacher.

4.1.5 Summary of results related to research question one

The results of the analysis of the data indicate that the ways that the teacher and students enacted the principles of productive disciplinary engagement were identifiable. Through the coding of verbatim transcripts of each classroom lesson, the participation pattern of the students and teacher became apparent. For example, the turn-taking pattern was often initiated by the teacher but followed by several consecutive student turns. Unlike the traditional I-R-E (Mehan, 1979) pattern, students' assumption of authority and accountability was apparent through peer evaluation of ideas, peers asking questions of each other, and by making their thinking public. As students voiced their thinking, others often made comments; adding to their thinking process or disagreeing. Critiquing the reasoning of peers was widely apparent. As students developed a sense of agency, they assumed some of the roles that are traditionally held by the teacher.

The teacher supported the students' assumption of both authority and accountability through the assumption of the *teacher as partner stance* and by offering choices to students. Offering choices and the teacher-as-partner stance worked together to offer students power: making them decision-makers while back grounding the authority of the teacher. Together, these

two pedagogical features helped to create a level of symmetry in the social configuration of the classroom. Offering choices related to the ways of approaching tasks, making connections, bringing prior knowledge and experiences, and using resources allowed students to further their decision-making capacity. The distribution of authority was apparent in the particular way the teacher invited student to notice features of mathematical information in whole group discussions. Through the invitation to notice, students were necessarily placed in a position as an active participant; challenged to sort through the visual cues; selecting and sorting information in an effort to identify particular mathematical features among competing bits of mathematical information. Features *noticed* by one student were not necessarily the same as the next student. This *noticing* served both an individual and a community function: as both the individual and the students sought to identify the pattern or feature as the original author viewed it. As times peers redirected the thinking of the original author, while at other times the author publically revised his thinking after listening to peer ideas. Widely apparent, the teacher positioned students as authors in each lesson, identifying them with their ideas. In addition, the use of Accountable Talk moves acted to probe student understanding and generate discussion. Through the use of specific pedagogical moves, she has positioned students as stakeholders and independent, capable, thinkers.

The strategic choice to include a large portion of tasks of high-cognitive demand offered students the opportunity to air uncertainty and to make public the construction of meaning among representations. Students persevered in problem solving in order to arrive at solutions that were accepted by their peers. The use of resources aided students in addressing the tasks. A wide variety of resources were used by students including material, ideational, and relational

resources. The provision of resources made the tasks doable for students and aided in the capacity to draw connections among representations.

The analysis of data related to research question one suggests that the principles of productive disciplinary engagement were present in the classroom over the fifteen days of instruction. A set of patterns that resulted from the analysis indicate that the principles of productive disciplinary engagement may be used in the design of a learning environment. The patterns that emerged with regard to the principles of productive disciplinary engagement include the following:

- Offering students power and making them decision makers through the use of Accountable Talk, offering choices, implementation of the teacher-as-partner stance, and positioning students as authors resulted in students exhibiting the capacity to critique the reasoning of peers and demonstrating the intellectual courage to hold peers accountable. Apparent in every lesson were students assuming roles that are traditionally held by teachers.
- The high-percentage of mathematical tasks of high-cognitive demand that were chosen contributed to student uncertainty that was apparent in every class. In addition, the task chosen contributed to students' need to explicate their mathematical thinking; providing an authentic need for classroom discourse.
- Material, relational, and ideational resources, carefully considered in lesson planning and enactment, were apparent in the classroom instruction and contributed to student access to the tasks offered to students.

4.2 RESULTS RELATED TO RESEARCH QUESTION 2

4.2.1 The work of the teacher

The results related to question one, indicate that the principles of productive disciplinary engagement were enacted by the students and the teacher. In reporting those results, much of the work of the teacher was reported. However, research question two points to the work of the teacher specifically; focusing attention on some of her efforts and thinking that was not apparent in the results related the first question.

Research question two addresses the work of the teacher. As a reminder, the question is:

- A) What work is required of the teacher in order to translate the principles of productive disciplinary engagement into practice?
- B) What challenges and successes does the teacher encounter along the way?

I will attempt to answer these questions using the data that includes lesson plans, reflections that were completed immediately following each day of classroom instruction, transcriptions of classroom lessons, and the tasks utilized. It is beyond the scope of one document to discuss all the elements of teacher decision making during planning, enactment, and lesson reflections. Therefore, through the examination of the plans and reflections, I was able to identify reasons for inconsistencies in the planning/enactment of the lessons that contribute to understanding the work of the teacher. In addition, the results of the second portion of the student questionnaire contributed information related to the establishment of social practices within the classroom that are instrumental in implementing the principles of productive disciplinary engagement.

4.2.1.1 Planning, enactment, reflection

Planning for the instruction is perhaps the most important task of the teacher who is interested in implementing the principles of productive disciplinary engagement; for it is this plan that defines the journey that is the lesson. With the learning objective, principles of productive disciplinary engagement, CCSS-M, Mathematical Practices, and state standards as a starting point, I constructed plans on a weekly basis. In most cases, plans were developed using the Connected Mathematics curriculum materials (Lappan et.al, 2014) as a guide. However, I did make some significant changes to the curriculum as it was written. The reasons that I made the changes and the changes themselves are detailed in the following section. Actual lesson plans are attached as Appendices J through T. Because the first page for each of the lesson plans was identical, I have included only page 2 for each lesson. Page one included the standards and practices, applicable to all of the lessons. At the bottom of each plan I have embedded the lesson reflection that I wrote directly following instruction. As I discuss reasons for decisions I made, I will refer to reflections on the day's lesson, if they are relevant. It would be beyond the scope of this work to discuss every change made or every decision during the course of each lesson. The addition of selected tasks, the inclusion of resources outside those available in the Connected Mathematics curriculum, and in-the-moment decisions related to the way to spend time were three decisions that I made as I considered what I thought students needed at the time, based on the information that I gathered from small and large group discussions

Among the decisions that I made during the planning process was to include additional tasks that would qualify as doing mathematics (Stein, Smith, Henningsen, & Silver, 2000). These tasks were added on days 1,2, and 5. The majority of tasks within the Connected Mathematics curriculum were high-level (procedures with connections) but were not as

challenging as some I had used previously. The tasks that I added were intended to cause students to think deeply regarding the concepts related to probability and to make connections among them. Hence, I added three tasks during the course of the fifteen lessons.

Among the tasks added was the marbles task (Table 4.20), adapted from the QUASAR study (Silver, Smith, & Nelson, 1995). This task and the anticipated student solutions are well documented by Smith & Stein, 2011. In fact the solutions that my students produced were consistent with my expectations and are shown in Appendix X. I anticipated a variety of representations on which my lesson depended. I had used this task with students in previous years at another school, so I was very aware of the potential for making connections among representations that the task offered. I modified the task to include the need for each pair of students to use two representations in their solutions, based on what I thought students understood about proportional reasoning from prior units, earlier in the year. As my reflection indicates, that was a good decision, because student pairs arrived at one solution easily, albeit not using the same representation. What caused the productive struggle for students was the need to utilize two representations. I was interested in student pairs making connections among representations and engaging in a challenge prior to the whole group discussion.

I find it to be interesting that at the time I wrote the reflection, I didn't consider the discussion to be among our most robust. I stated that I was tired, and wondered whether that had affected the discussion. However, when I watch the classroom video and read the transcript, I would not arrive at that conclusion at all! Students were eager to engage and discussion was rich. I had selected student work strategically for student presentations and made a concerted effort to ask probing questions aimed at making connections among the presentations. The order of presentations included Bryce and Lauren, who had used the percent of blue marbles in the

total bag to arrive at their conclusion. The second group to present was Madia and Henriët. They utilized unit rate; number of total marbles for each blue marble. Finally, Ed and Estelle presented their graph; illustrating the proportion of blue marbles to red marbles. The discussion encompassed nearly a full class period, highlighting key concepts embedded in the task. In addition, based on the discussion I recognized which students could make the connection between proportions, probability, and linear relations. So the task served as a formative assessment in addition to a cognitively challenging task for students.

Two other tasks were interjected by me that qualify as doing mathematics: the orally assigned graphing task on the first day, and the development of a definition for experimental and theoretical probability on day 2. The graphing task on the first day was intended to cause students to move from concrete to more abstract thinking. They had flipped coins on the first day and knew that half of the flips should be each heads/tails, but had not achieved those results exactly with a sample of thirty flips. The question that I asked was,

“Some groups are done, some are not, while you are waiting, just take a piece of notebook paper and just sketch this for me... Percent of heads...just a trend. Folks, percent of heads (pointing to the vertical axis of a sketch on the board), and number of tosses (pointing to the horizontal axis). This is a small number of tosses, and this is a larger number of tosses as you go to the right. As this goes on, how is this going to look. Is it a straight line? Is it going to be this way (indicating a negative slope)? Will it be jagged? (indicating with a marker), What’s it going to look like if you just kind of guessed what a graph of this data would look like. Collecting more data, more tosses, might change how this looks. Just theorize with the graph...what this might look like.”

I thought this exercise would challenge them to consider that the experimental data gathered over many trials approached the theoretical probability. In the discussion the previous day, it had been apparent that no students had recognized that more coin tosses provided a better predictor of the frequency than one sample. I wanted to draw attention to that concept.

The lessons in the Connected Mathematics (Lappan et.al, 2014) attempts to draw students' attention to the idea of the Law of Large Numbers through the repeated combination of student data into one large group of data for coin tossing, cup flipping, etc. My students engaged in these tasks as prescribed. However, because I had not heard evidence that they understood, I augmented that experience with a simulation using NCTM Core Tools (www.nctm.org) during day 4. Using the simulation, students watched as I used the tool on the Smartboard. Students engaged in a teacher-led, discussion as they watched the change in frequency graph as the sample size increased.

Teacher: (sitting at the computer with the smart board on – about to bring up a simulation from Core Tools on the NCTM website) So I want to show you a little simulation...this is pretty cool. So we're going to pick the die and we're going to conduct ...ten experiments. So what would you expect to happen? You know that one out of 6 times...are they equally likely? To land on a 1 , a 2, a 3, a 4, a 5, a 6, are they ..is that equally likely when you flip a dice? No?

Students: several voices say yes

Teacher: It's equally likely right? You could land on a 2 just as well as a 3. So at the end of things you'd expect to have, theoretically, you'd expect to have the same number of 2s rolled as 3s , right? K, so let's watch what happens. (The simulation shows the die on the top and a bar graph indicating frequency on the bottom). Here's a graph of the number of 1s , 2s , 3s , 4s...What do you see here? Are they equal?

Students: shake heads and say no

Teacher: Not very equal. So let's do a couple more flips of this dice. (using simulator) What do you notice happened?

Student: They're more equal.

Teacher: More equal... (still the bar graph indicates unequal proportions) How about if we do 500?

Students: some

Teacher : Somewhat equal..How about if we do 5000?

Students: (inaudible)

Teacher: Pretty close to equal. So what could you say from that? Can you make a generalization? This is the experimental probability, right? What can you say about the theoretical probability if this is the experimental probability? Talk in your groups for a second. See if you can generalize something from that. The sample size got bigger, bigger, bigger, right? Related to the theoretical probability, what happened there?
(students talk in small group while teacher joins each group briefly)

The lesson plan for day 4 includes no mention of using Core Tools, although the reflection does. I had planned for the fourth day of instruction without every talking to students about probability before. Hence I had no idea of their conceptions regarding the topic when the lesson plan was developed. What I thought I was going to do seemed inappropriate for the students as the first three days of instruction unfolded. Therefore, I drew upon a resource that I thought could help students develop a deeper conceptual understanding and included it in the lesson.

An event map of the lesson, shown in Appendix H further verifies the dichotomy between the lesson plan (Appendix M) and the enacted lesson. The event map indicates the frequent transitions between whole group and small group work within the classroom, that is not indicated in the lesson plans. The reason for transition to small group is an in-the-moment decision based on many factors . However, among the reasons to transition to small group for short segments is to offer all students an opportunity to engage in thinking and discussing about the question at hand. By engaging in discussion with students in small groups I also gather information about what students are thinking.

In addition, the event map indicates the topically related segments (TRS) that are the subject of discussion among the teacher/students. This lesson, typical of most of the lessons in this unit of study, consists of four separate but related topics. The lesson begins with a topic that carried into this lesson from the prior day, followed by students attempting to define

experimental and theoretical probability. This discussion is followed by the Core Tools demonstration, intended to further inform students' conceptual understanding related to experimental and theoretical probability, and then closing with a partner quiz. Although I had intended to follow my plans when they were constructed five days earlier, the enactment of the lesson had little resemblance to the plan. Not all lessons were this divergent from the plan. For example, day 10 lesson plan includes at the beginning of the lesson the review of four homework problems. In fact, homework review problem number four was the source of discussion for the entire class period. The topic was consistent with the plan, but I made a decision to allow discussion of the task to continue based on the student needs at that moment. An event map of the class, shown in Appendix U, illustrates the flow of the class, managed by the teacher, during enactment of the lesson. The accompanying transcription for the whole group discussion ranging from 22:14 through 40:03 is noted in Table 4.22, discussed earlier. The rich discussion resulting from the in-the-moment decision to allow the class to discuss homework problem 4 resulted in a high degree of student participation and the exploration of an area model as a tool for use in compound events.

The addition of selected tasks, the inclusion of resources outside those available in the Connected Mathematics curriculum, and in-the-moment decisions related to the way to spend time were three decisions that I made as I considered what I thought students needed at the time, based on the information that I gathered from small and large group discussions. These decisions were instrumental in implementing the principles of productive disciplinary engagement. The task selection encouraged uncertainty and offered students a rich topic of discussion, essential elements in problematizing. Adding Core Tools as a resource for students was one way that I tried to offer access to students regarding the conceptual understanding of the

Law of Large numbers. The decision to spend additional time on a problem, beyond what I had planned to spend also was a necessary resource to provide at the time.

4.2.1.2 Interpreting students' mathematical thinking

Interpreting the substance of students' mathematical thinking and reasoning and then improvising subsequent instruction in response to various elements of their thinking is what Black & William (2007) call formative assessment, and what Ball (1993, p. 374) referred to as the "twin imperatives of responsiveness and responsibility". In a class discussion, interpreting student mathematical thinking demands that teachers listen carefully to student reasoning and respond in a way that moves students toward more sophisticated conceptual understanding. It is related to what Stein, Engle, Smith, & Hughes (2008) refer to as monitoring; a process of paying attention to the thinking of students during the actual lesson as they work individually or collectively on a particular task. This attention to student thinking by the teacher is among the features of an environment that is conducive to the implementation of the principles of productive disciplinary engagement, for it drives the communication pattern and social interactions on which learning depends.

Attending to student thinking during the enactment of a lesson, demands that teachers sort the many sources of information that are available at the moment, and choose among them to determine which deserve attention. In order to attend to student thinking, a teacher must first "tune in" to it. This attention is what is currently referred to in the mathematical literature as noticing. I have adopted the definition of noticing used by Sherin, Russ, & Colestock, 2008. That is, noticing is exploring what a teacher attends to as well as what the teacher decides not to attend to. It includes the filtering of activity as well as the teacher's interpretation of that activity.

In the current study, lesson reflections that were composed immediately following the lesson drew attention to my awareness of the ways that students were thinking. Among the patterns included in the lesson reflections was the word, “notice”. I admit that I was surprised by this finding; not realizing the extent to which I was “noticing” what the students were thinking. In fact, the word “notice” was written ten times during the fifteen paragraphs that constitute the lesson reflections. During the coding process of this project, I had coded student discourse for “noticing”; carefully examining the transcripts for the patterns and mathematical content to which they attended. I realized that my report regarding what the students noticed, depended on what I noticed. The analysis of these lesson reflections provided an opportunity for the unveiling of the lens through which I see their work and my role in it. The way that “notice” has been used in the lesson reflections follow. I have labeled each with a letter for later reference.

- A) “Ed noticed that the percentage should hover around 50%, but at first many didn’t even notice that.” (day 1)
- B) “Some students definitely noticed the value of a large sample size via the provided student graph while others are still working on it.” (day 2)
- C) “ I notice that students are beginning to use the word “outcome” and several saw the relationship with proportional reasoning.” (day 2)
- D) “I noticed they are trying to provide evidence of their thinking, even without prompting.” (day 3)
- E) “Students struggled to generalize in words what they noticed as the sample size increased due to limited familiarity with vocabulary, I think.” (day 4)
- F) “Second, I wondered if kids noticed that probability had to be less than or equal to one.” (day 4)

G) “One student did an amazing job at the end of the class in generalizing the pattern; noticing that the number of pennies would always be odd for two colors of gumballs.”

(day 8)

H) “I’m sure they noticed that I valued that capacity.” (day 8)

I) “I wonder if tomorrow, students will notice they are multiplying the probability of one bucket with the other,” (day 9)

J) “A few noticed that the likelihood of getting two points was 36% but no one said anything about multiplying .60 times .60 to get it.” (day 9)

Most of these examples point to information gathered related to student thinking that might influence subsequent instruction. Quote J, for example, indicates attention to helping students draw a connection between the area model and the multiplicative procedure that might be used to find the probability of a compound event. Students, working on an area model to represent a compound event, in Part C of Problem 4.1 (Appendix I) were not linking the percent they had determined from the area model to a multiplicative procedure of length times width. I considered that idea for later lesson integration.

Likewise, quotes E and G focused on student capacity to generalize solutions. Quote E refers to students’ response to the teacher’s prompt for a generalization regarding the relationship between theoretical and experimental probability after they had watched a Core Tools simulation that demonstrated the change in experimental probability with large sample sizes. Students struggled with the generalization despite witnessing the change in experimental probability, a noteworthy event to the teacher. Quote G refers to an uncommonly insightful generalization by Ute following class work on the Sticky Gum problem (Appendix I). Ute had recognized that the

number of pennies necessary to ensure two similar-color gumballs would always be an odd number.

Quote A and B point to the apparent disparity in students' capacity to notice the importance of sample size with regard to probability. This observation led to a lesson that included a computer simulation on day 4, using Core Tools wherein sample size could be increased easily and the effect regarding experimental probability was documented via changes in data on the screen.

Quote C reflects my view of the value of talking like a mathematician. There are many instances in the transcripts where I interject mathematical vocabulary in revoicing a student's contribution. In addition quote C is indicative of the importance of relational understanding; knowing what to do and why. It reflects my belief that integrating new ideas into a rich web of concepts through the use of multiple representations improves student retention (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Quote C indicates an instance of feeling successful in students' use of vocabulary.

Another major theme among these quotes is equity. I am challenged to encourage the participation of some reluctant students who are afraid of intellectual risk taking and so are quiet. Understanding that there are some quiet, but engaged, learners I am more comfortable that I know students are engaged when I have the tangible evidence that classroom discourse provides. I don't want attention-seekers to diminish the opportunity for those students who may be tentative but want to speak. Hence, several of these reflections point to that idea.

K) "I am challenged to diminish the talking of some and encourage talking in others." (day 13)

L) “Henry is still not taking risks. I think it is time for a formative assessment to see if all are understanding.”

M) “ I’m quite sure that the whole class did not follow his thinking but that is ok with me.

I feel that the students who are most capable of thinking abstractly had the opportunity and encouragement to do it, and those that are developing that capacity had someone to model it for them.” (day 8)

These quotes point to equity with regard to opportunity to speak, but also with regard to the value of more capable students modeling abstract thinking for those developing the skill. Quote M indicates that I value opportunities for abstract thinking, and I believe it is a skill to be developed, not an innate ability. Quote L points to the value of formative assessment in both oral and written forms so that instructional modifications may be made so that all students make progress toward conceptual understanding.

Quote D, focuses attention on a point of success. Students providing evidence of their thinking without prompting is celebrated in this quote, obviously a point of attention for me in lessons prior to this lesson. Interest in evidence of their thinking indicates that I value the logic that students use to develop conjectures and deem them to be capable of explaining it.

The reflection of the meaning of these lesson reflections is a rich source of information related to the beliefs and habits of mind that guide my work of teaching. Through the analysis of these reflections I have established for the reader, and reminded myself, of the beliefs that undergird the enactment of the principles of productive disciplinary engagement in my classroom. Why is equity, relational understanding, adapting teaching to student understanding, and students’ capacity to provide evidence of their thinking important to enacting the principles of productive disciplinary engagement (authority, accountability, problematizing, and

resources)? They reflect my beliefs about what it means to know and do mathematics and about the ways that students make sense of mathematics. What I notice, depends on what I believe to be important.

Closely related to noticing, monitoring (Stein, Engle, Smith, & Hughes, 2008) refers to the teacher paying attention to student thinking as they work toward the solution of a task. Anticipating, monitoring, selecting, sequencing, and connecting are commonly referred to as the 5 Practices for Orchestrating Discussion (Smith & Stein, 2011). Each of these practices is evident in the teacher's work. Anticipating was apparent in lesson plans in the form of preplanned questions in every lesson. Considering carefully what questions to ask and what solutions were likely to be completed by students, were done together. Anticipating student solutions was also apparent in the day 5 lesson plan in the form of mentioning using a monitoring tool, a practice often utilized. Considering what solutions were likely, then keeping track of student thinking (monitoring) during enactment of the task contributed to the enactment of the principles of productive disciplinary engagement. Finally, choosing what solutions would be discussed during whole class discussion was a critical component to the enactment. Noticing student solutions that would generate discussion (such as Estelle's tree diagram) and ordering solutions to encourage the connections between probability and proportional reasoning were evident in nearly every lesson plan. Preplanning how I would make the connections among representations using student work, allowed me to relax and manage the talk instead of focusing my thinking on the mathematics while the discussion was proceeding (Stein, Engle, Smith, & Hughes, 2000).

4.2.1.3 Developing questions that elicit evidence of learning

Questioning has been identified as a critical and challenging part of a teacher's work. Boaler & Brodie (2004) noted that, "the act of asking a good question is cognitively demanding; requires considerable pedagogical content knowledge, and necessitates that teachers know their students well" (p. 773). They explain the importance of questions in shaping the nature and flow of classroom discussion. The Boaler & Brodie (2004) study resulted in a tool that is useful for categorizing teacher questions; used in this study (Appendix B). Sharing the definition of a question, used by Boaler & Brodie (2004), every teacher question in thirteen lessons was coded. (The final class of the unit was an assessment for the entire class, and as such did not include questions of a mathematical nature. The 12th class included a technical difficulty wherein no audio was captured.) I, like Boaler & Brodie (2004), chose to include utterances that had both the form and function of a question. That is, I excluded statements that sounded like a question but didn't function as such. For example, "Would you like to share your thinking?" would not be coded. In addition, questions needed to be mathematical in nature. I would exclude the question, "Do you have your homework?". If a question were repeated, I counted it only once.

Results of the coding are indicated in Figure 4.13. and in Table 4.7. A summary of questions asked each day of instruction are illustrated in Appendix Y. The data illustrates the consistently high proportions of questions classified as probing, generating discussion, and exploring mathematical meaning.

Table 10. Questions by Type (adapted from Boaler & Brodie, 2004)

Gathering information	55
Inserting Terminology	15
Exploring mathematical meaning	88
Probing	118
Generating Discussion	140
Linking/applying	8
Extending Thinking	33
Orienting/focusing	22
Establishing Context	3
Total questions	482

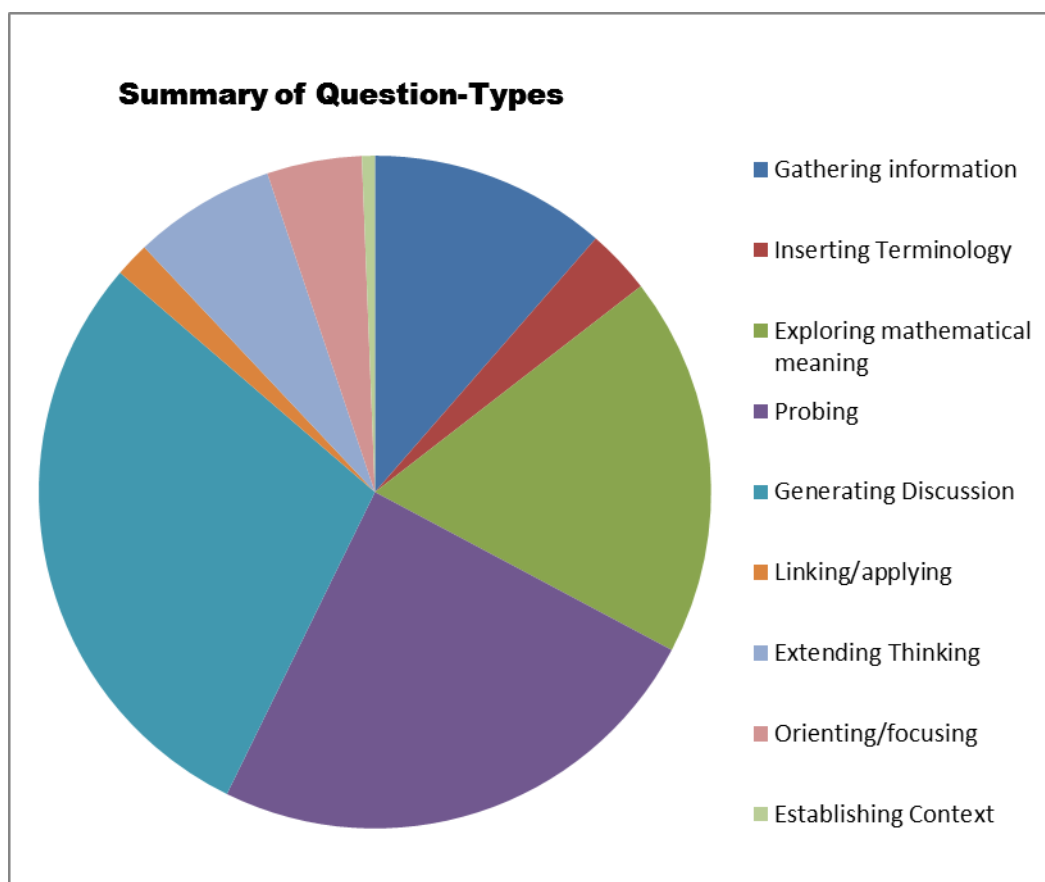


Figure 14. Questions by Type (adapted from Boaler & Brodie, 2004)

Roughly two-thirds of the questions in the sample were categorized as generating discussion, probing, or exploring mathematical meaning. With the remaining third comprising

the other categories. Generating discussion-type questions accounted for approximately one-third of the questions. This is not surprising, based on the teacher's purposeful attempt to encourage classroom discussion, a necessary element for the enactment of the principles of productive disciplinary engagement. The question-type ranking second in use was probing. Making students accountable to themselves, the community, and the discipline requires probing of student thinking. Thinking deeply to encourage conceptual understanding was among the goals of instruction. Gathering information, the question-type most prominent in traditional classrooms, accounted for 11% of the questions in the study. The large variety of questions is consistent with finding by Hiebert and Wearne (1993) wherein teachers in "alternative" classrooms asked a larger range of questions and more questions that required explanation and analysis than did teachers in "traditional" classrooms.

The lesson plans illuminate the consideration of questions prior to instruction by the teacher. The section entitled, "assessing and advancing questions" includes questions aimed at eliciting student mathematical thinking and encouraging the students to think deeply about the mathematical content. Although I often did not use the questions exactly as they were planned, planning the questions served the purpose of thinking deeply during the planning process. Through the consideration of specific questions, I rehearsed a way to phrase a question that encouraged students to talk about their thinking. Using this technique helped me to avoid questions that might be answered with a single number or yes/no answer.

The answers related to several questions on the student questionnaire further validate the use of teacher questions in probing student's mathematical thinking. The question asks, "How much do you agree with the following statements about your teacher in your Math class?" (circle the answer that reflects your opinion). Question 3 is "Often requires me to explain my answers".

Although the question does not delineate between oral and written explanations, of the 19 students in the class, 14 students responded strongly agree, and 5 responded agree. Question 8 also addresses teacher questions. It also asks for a response ranging from strongly disagree to strongly agree related to the phrase, “Wants us to become better thinkers, not just memorize things.” Thirteen students responded strongly agree and 6 responded agree on this question. Although several factors may have contributed to student’s opinion on this question, I propose that teacher questioning was among the features of the environment that contributed to that response. Likewise, question 2 on the survey states, “Encourages students to share their ideas about things we are studying in class”. One of the ways that students are encouraged to share is through teacher questioning; specifically those that generate discussion or ask students to explain their thinking. Likewise, number 4 of the questionnaire requests a response to, “Encourages us to consider different solutions or points of view.” Through the generation of discussion, students not only share their own ideas but become active in considering the solutions of other students. Teacher questioning contributes to responses on the student questionnaire.

4.2.1.4 Establishing social practices –results of the student questionnaire

The student questionnaire addresses students’ perceptions regarding many of the social practices that contribute to the definition of the classroom norms. Who is free to do what, who assumes responsibility for certain things, who explains and who listens at what times, are just a few of the social practices that must be consciously established and which students are acutely aware. The survey questions and the results are shown in Table 4.8. Numbers below each response indicate the number of students who independently voted for that response.

Table 11. Student Survey and Results

How much do you agree with the following statements about your teacher in your Math class: My teacher: (circle the answer that reflects your opinion)

1.	Often connects what I am learning to life outside of the classroom.			
Strongly disagree	Disagree	Agree	Strongly agree	
0		5	14	
2.	Encourages students to share their ideas about things we are studying in class.			
Strongly disagree	Disagree	Agree	Strongly agree	
0	0	5	14	
3.	Often requires me to explain my answers.			
Strongly disagree	Disagree	Agree	Strongly agree	
0	1	6	13	
4.	Encourages us to consider different solutions or points of view.			
Strongly disagree	Disagree	Agree	Strongly agree	
0	1	9	9	
5.	Encourages students to stop working when the work gets hard.			
Strongly disagree	Disagree	Agree	Strongly agree	
13	4	1	1	
6.	Expects us to work together to solve problems.			
Strongly disagree	Disagree	Agree	Strongly agree	
0	0	12	7	
7.	Gives us work in class that is challenging.			
Strongly disagree	Disagree	Agree	Strongly agree	
0	2	11	5	
8.	Wants us to become better thinkers, not just memorize things.			
Strongly disagree	Disagree	Agree	Strongly agree	
0	0	6	13	

Table 11 (continued)

9.	Wants us to discuss possible solutions to problems with other students.			
Strongly disagree	Disagree	Agree	Strongly agree	
1	0	7	11	
10.	Rarely asks students to show their work on the board or document projector			
Strongly disagree	Disagree	Agree	Strongly agree	
12	4	1	2	
11.	Makes resources (graph paper, spinners, books) available to us in case we need it.			
Strongly disagree	Disagree	Agree	Strongly agree	
0	0	6	13	
How often does this happen in your mathematics lessons?				
a)The teacher shows us how to do mathematics problems.				
never	once in a while	usually	always	
11	5	1	2	
b) We copy notes from the board.				
never	once in a while	usually	always	
7	12	0	0	
c) We work on mathematics tasks in small groups				
never	once in a while	usually	always	
0	0	8	11	
d) Students use the board or document projector				
never	once in a while	usually	always	
0	2	6	11	
Table 11.(continued) Student Survey and Results				
e) The teacher uses the board or document projector				
never	once in a while	usually	always	
0	4	8	7	
f) If we don't know how to solve a difficult problem we ask other students for help.				
never	once in a while	usually	always	
0	3	7	9	
g) If we don't know how to solve a difficult problem we ask the teacher for help.				
never	once in a while	usually	always	
1	6	8	3	

Table 11 (continued)

h) Desks are organized in rows so that we can work independently.			
never	once in a while	usually	always
10	8	0	0
i) During student presentations, we ask questions of each other if we don't understand what he is explaining.			
never	once in a while	usually	always
0	2	6	11
j) I feel free to invent my own way to solve a mathematics problem.			
never	once in a while	usually	always
3	2	9	6
k) Students read and work from the textbook while the teacher talks about it.			
never	once in a while	usually	always
11	7	1	0
12. Students listen while the teacher explains rules and definitions.			
never	once in a while	usually	always
6	6	2	6
m) Students, together with the teacher, decide whether an answer is correct.			
never	once in a while	usually	always
3	1	5	1
n) The teacher decides whether an answer is correct.			
never	once in a while	usually	always
2	7	8	2

While I have referenced responses from the first portion of the survey in the previous sections, I draw attention here to the results of the second section, “How often does this happen in your mathematics lessons?” in Table F and Table 4.9. These questions address the classroom norms on a large grain size, focusing attention on what a naïve observer might “see”. Students largely agree, based on the frequency of responses, on the following features of the classroom. Specifically, at least 80% of students agree or strongly agree that these features are present.

Table 12. Features of the Classroom described by the Student Questionnaire results

<u>Feature</u>	<u>Question Referenced</u>	<u>Related Principle</u>
The teacher does not show students how to do mathematics problems	a	authority, accountability, problematizing
They usually don't copy notes from the board	b	authority, problematizing
Students work on tasks in small groups	c	problematizing, resources
Both the teacher and students use the document projector	d,e	authority, accountability
Desks are rarely organized in rows	h	resources, authority
During student presentations, peers ask questions of each other	i	accountability, authority
Students rarely work from a textbook	k	problematizing
Students and teacher decide when an answer is correct	m	authority
accountability		

The conclusion that one might draw related to questions f, and g (dealing with who students ask for help) is less clear. Sixteen students (84%) indicated that if they are uncertain about the way to solve a problem they always or usually ask a peer (f). Eleven students (57%) indicated that if they are uncertain they always or usually ask the teacher (g). Perhaps a subset of students are expressing that they ask both peers and the teacher for help if they are uncertain.

Although these features would not, alone, create an environment that is conducive to the enactment of the principles of productive disciplinary engagement, they are integral to the

foundation necessary to do so. Examining the features and concurrently considering the principle addressed via that classroom feature draws attention to the interconnected nature of each principle.

4.2.2 Challenges and successes

The most significant challenge for a teacher who is engaged in enacting the principles of productive disciplinary engagement is the redefining of success for students. Students arrive, having had experience in the way to succeed in mathematics class. They know the routines and expectations of a traditional classroom. They understand how to get good grades, when to stay quiet, who the authority figure is, and what the routines include. The very best students were most confident in the way to succeed. In many cases, they had been successful for their entire school career, participating in a traditional classroom. Convincing these students and their parents that new rules, routines, and norms applied and teaching them the way to participate in a non-traditional class, was not easy work. For months, students were reluctant to participate in the conversation necessary to define the class. It took consistent teaching about the way to participate; when to ask questions of peers, providing example questions, generating discussion through my own questions before students began to participate. Then there was the problem of grading. They were used to an assessment with many questions that were all replicas of problems completed in class. Again, expectations needed to be redefined. My estimate is that it took six weeks for the first students to change their ideas regarding success in the mathematics class, while others took at least twice as long, and a few resigned themselves to participating but not loving it. Without the full support of the district administrative, I could not have succeeded in my journey to enact these principles.

However difficult it was to get going, I felt success by the end of the school year. Most students were eagerly participating and assuming responsibility for their own learning as demonstrated in this work. Finally, the most significant marker of success came in the form of a student note shown in Figure 4.14. Students, themselves, realized the change that they had endured. They recognized that perseverance was a necessary part of learning math. That made all of my effort worth the challenge.



Figure 15. Student Note

5.0 DISCUSSION

In this chapter, a discussion of what can be learned from this investigation and how the findings can inform the design of norms, structures, and classroom features that combine to form a learning environment is presented. The chapter begins by describing the importance of the study including a discussion of the ways in which this investigation contributes to the knowledge base of research related to the decomposition of effective teacher practices and the identification of related student behaviors that contribute to the design of a learning environment. Next, possible explanations for the results of the study are presented. The chapter closes with concluding remarks and suggestions for further research.

5.1 IMPORTANCE OF THE STUDY: USING THE PRINCIPLES OF PRODUCTIVE DISCIPLINARY ENGAGEMENT AS A TOOL FOR CLASSROOM DESIGN

The purpose of this study was twofold: 1) to describe the ways that the principles of productive disciplinary engagement were evident in the instructional practices implemented by the teacher and enacted by the students, and 2) to explicate the work of the teacher in translating the principles into practice. This phenomenon was examined using the researcher as a teacher in a seventh-grade mathematics classroom, collecting video data, lesson plans, teacher reflections,

mathematical tasks, and a student questionnaire. The study's focus was student reasoning and communication, consistent with the CCSS-M Standards for Mathematical Practice.

The results of this study are important to teaching and learning because key features of innovative instructional environments that have been captured by Engle & Conant's (2002) principles of productive disciplinary engagement, originally offered as a theoretical framework for designing supportive learning environments, have been used as a practical tool for the design of a learning environment. The study provides insight into the extent to which the framework is useful for this purpose. Research that has been published since the original work that introduced the principles of productive disciplinary engagement has been extensive (Meyer, 2013; Kelly, 2013; Venturini, & Amade-Escot, ., 2013; Windschitl & Thompson, 2006; Gresalfi, Hand, and Hodge, 2006). In addition, Engle (2011) reviewed seventeen case studies that used the principles of productive disciplinary engagement. The work to date suggests that there is some consensus within the research community that the principles of productive disciplinary engagement capture a wide array of respected educational innovations developed over the past twenty years (Forman, Engle, Venturini, & Ford, 2013). However, the work to date provides little guidance to teachers or teacher educators regarding the ways to operationalize these ideas in the classroom. Results of the study reported herein serve a practical purpose in guiding others interested in the design of learning environments, and extends the work of Engle & Conant (2002) by employing the framework as a tool for the design of learning environments by practitioners. The study may be of particular interest at the present time as teachers struggle to enact mathematics instruction consistent with the eight Standards for Mathematical Practice, an integral component of successful implementation of the CCSS-M.

In addition, this work extends the work of Engle & Conant (2002) by attending to both the ways the teacher and students enacts the principles. The original work, focused only on the students, placed the work of the teacher in the background. This study provides a compelling argument for the reflexive relationship between the teacher and student behaviors and the value in considering both simultaneously when considering these principles.

The results of this study provide evidence that the principles of productive disciplinary engagement may be useful in the design of a learning environment. Further it describes the instructional practices of the teacher and ways that students engaged in the classroom when these principles undergirded the creation of the environment. Table 5.1 summarizes the results obtained from this study.

Currently, teachers across the country are struggling to enact the Standards for Mathematical Practice in the CCSS-M. Although the CCSS-M helps to focus and clarify intended outcomes, it does not prescribe the actions, practices, programs, or policies for successful implementation. The principles of productive disciplinary engagement inform those who intend to teach in concert with the CCSS-M Standards for Mathematical Practice. Specifically, the first three practices: make sense of problems and persevere in solving them; reason abstractly and quantitatively; and construct viable arguments and critique the reasoning of others, focus on making sense of problems and solutions through the process of logical explanation as well through probing the understanding of others. As students construct arguments, identify correspondences among approaches, and explore the truth of conjectures they are both enacting the CCSS-M Mathematical Practices and enacting the principles of productive disciplinary engagement. An environment that is supportive of the CCSS-M and the principles of productive disciplinary engagement offers an opportunity for developing a shared

experience that uses differences in student thinking as a tool for productive collective work (Boaler & Staples, 2008; Hufferd-Ackles, Fuson, & Sherin, 2004). Table 5.2 illustrates several parallel behaviors, identified in this study, that are examples of both the CCSS-M Standards for Mathematical Practice and the principles of productive disciplinary engagement. This list is not meant to be exhaustive; only to provide the reader with a sense of the commonality in student behaviors described in each document.

Table 13. Summary of Results

Ways the Teacher and Students Enacted the Principle of Authority
The participation pattern did not follow the traditional IRE pattern. Although the teacher frequently initiated discussion, several student -turns often followed. The attention to pedagogical moves such as the use of Accountable Talk and questioning contributed to the development of the desired participation pattern.
Instances of offering choices and the implementation of the teacher-as-partner stance were found together. The teacher's posture and the attention to student choices offered students power and provided opportunities for decision making.
As students developed a sense of agency, they assumed some of the roles traditionally held by the teacher including critiquing the reasoning of peers .
The opportunity to notice features of mathematical information during whole group presentations served to draw attention to students' own thought processes and to distinguish one student's thinking from another.
Positioning students as authors by publically identifying them with their own claims approaches, and explanations was widely apparent.
Ways the Teacher and Students Enacted the Principle of Accountability
Thinking was both a private and public event. Placing students in a position to publically revise their thinking allowed for thinking to be an individual and a community responsibility.
As they assumed roles that are traditionally held by the teacher, students demonstrated intellectual courage to hold peers accountable .
Use of Accountable Talk was one move used by the teacher to encourage students' accountability to the community, the discipline, and each other.

Table 13 (continued)

Ways the Teacher and Students Enacted the Principle of Problematizing
The mathematical tasks chosen by the teacher included a high percentage of tasks requiring a high cognitive demand, contributing to student uncertainty and the need for students to persevere in problem solving and explicate their mathematical reasoning.
Student uncertainty regarding the way to proceed in solution, or what to conclude, was apparent in verbatim transcripts in every lesson, and was accomplished by the choice of the mathematical task.
Ways the Teacher and Students Enacted the Principle of Resources
Material, relational, and ideational resources were carefully considered in lesson planning and enactment, and apparent during the classroom instruction.
The Work of the Teacher in Translating the Principles into Practice
The addition of selected tasks, the inclusion of resources outside those available in the curriculum, and the in-the-moment decisions related to the way to spend time were among the decisions made based on information gathered from small and large group discussions.
Attending to student thinking during the enactment of the lesson informed instructional decisions.
The approximately two-thirds of all teacher questions in the category of exploring mathematical meaning, generating discussion, or probing were evidence of the teacher's purposeful effort to generate discussion related to mathematical conceptual understanding.
The established social practices that contributed to the definition of classroom norms were defined by the student questionnaire.
The most significant challenge for a teacher who is engaged in enacting the principles of productive disciplinary engagement is redefining success for students. The feeling of success comes when students assume responsibility for their learning and contribute as a part of a learning community, once the principles are in place.

Table 14. Partial List of Student Behaviors Found in this Study

Student Behavior	CCSS-M, Mathematical Practice	Principles of Productive Disciplinary Engagement
As students developed a sense of agency, they assumed some of the roles traditionally held by the teacher including critiquing the reasoning of peers	Construct viable arguments and critique the reasoning of others	Authority
Positioning students as authors by publically identifying them with their own claims approaches, and explanations was widely apparent.	Construct viable arguments and critique the reasoning of others	Authority
The opportunity to notice features of mathematical information during whole group presentations served to draw attention to students' own thought processes and to distinguish one student's thinking from another.	Construct viable arguments and critique the reasoning of others. Reason abstractly and quantitatively.	Authority
The mathematical tasks chosen by the teacher included a high percentage of tasks requiring a high cognitive demand, contributing to student uncertainty and the need for students to persevere in problem solving and explicate their mathematical reasoning.	Persevere in problem solving. Reason abstractly and quantitatively.	Problematizing

Table 14 (continued)

Thinking was both a private and public event. Placing students in a position to publically revise their thinking allowed for thinking to be an individual and a community responsibility.	Construct viable arguments and critique the reasoning of others Reason abstractly and quantitatively.	Authority and Accountability
As they assumed roles that are traditionally held by the teacher, students demonstrated intellectual courage to hold peers accountable.	Construct viable arguments and critique the reasoning of others. Reason abstractly and quantitatively.	Authority and Accountability
Student uncertainty regarding the way to proceed in solution, or what to conclude, was apparent in verbatim transcripts in every lesson, and was accomplished by the choice of the mathematical task.	Persevere in problem solving.	Problematizing

The student behaviors described in Table 5.2 and in the preceding sections of this document are evidence of the teacher’s goal for students to participate in a very specific way; a goal that was defined well before the study began by the definitions of each of the principles of productive disciplinary engagement. The principles of productive disciplinary engagement were used as the tool that helped me to create the environment that supported the defined participation pattern. I argue that these principles may be used by teacher educators to help teachers redefine success for themselves. They may be utilized both as a framework for instruction- related practices that support the CCSS-M, as well as to help teachers establish new measures of success.

5.2 EXPLANATIONS OF THE RESULTS

This section offers explanations for the results obtained in this study. The enactment of principles of productive disciplinary engagement relies significantly on the view that mathematics is learned not by the transmission of knowledge, but rather by participating in a culture as part of a “social practice” (Lave & Wenger, 1991, p. 47). That is, mathematical learning entails both social and communicative activities in a supportive learning environment wherein the member of the community are engaged in “dense relations” of mutual engagement organized around something that matters to the group (Wenger, 1998, p. 74). The principles of productive disciplinary engagement were used as a framework for the design of that learning environment. What follows is an explanation for the reasons that these principles were useful for this purpose.

5.2.1 The mathematical task: A critical element of implementing the principles of productive disciplinary engagement

Each of the principles of productive disciplinary engagement (Engle & Conant, 2002) was embodied in the environment in numerous and interconnected ways. The selection of the mathematical task, an element of the principle of problematizing, was a critical element in the entire process of creating the learning environment, for it provided the foundation upon which the enactment of the other principles depended. Figure 5.1 illustrates the interconnected nature of the principles and the central role of the mathematical task.

Through the examination of the tasks and consideration of my goals related to the ways students would participate in the classes, I have become aware that task selection included more

than the criteria set forth in the Math Task Analysis Guide, as I had planned and reported in Chapter 3. In fact, upon reflection, in addition to the cognitive demand of the task, I also chose tasks considering the extent to which a robust classroom could be created. For example, as I chose the Caves Task for day 15, I recognized that providing four solution paths would allow me to encourage students to assume authority through a vote for one solution over the others. I have found that tasks which provide solution options typically afford the opportunity for robust classroom discussion. I enlarged the solutions and hung them on the front classroom wall in an effort to provide a resource for the students that further encouraged the discussion. During the class, students walked freely to the front of the class to adamantly oppose peer votes for the correct solution. The task selection was critical to the implementation of the principles of productive disciplinary engagement, and my commitment to the enactment of the principles also was critical to the task selection. In addition to the Caves task, tasks utilized on days 8, 9, and 10 shared a similar characteristic of encouraging students to align themselves with a particular solution. In so doing, students necessarily defended their positions.

The Caves task offered students the necessary complexity for sorting and classifying information. As a part of the solution process, it was apparent that different students noticed different features of the task. Those differences contributed to the variety of solution processes developed; offering students the opportunity for authorship. Differences in student focus also provided something worthy of discussion, as students worked cooperatively to solve the task. As each pair authored their own solutions, it was necessary for each to engage in considering what his partner noticed; placing students in a position to consider alternative ideas. In general, the tasks themselves offered students the opportunity to solve them in different ways, with a variety

of entry points. Depending on students' prior understanding, and the way they viewed the tasks, they proceeded toward solution in one of several possible ways.

The task also stimulated a sense of uncertainty in students. Because it was necessary for students to think deeply regarding how to proceed, the way to represent solutions, and what to conclude, they often expressed uncertainty to their peers and to me. The uncertainty, in turn, also provided an impetus for discussion. Students talked because they were placed in a position where talk was a necessary element of success. Further, middle school students are naturally social, and so this need for discussion was eagerly embraced. Peers, a relational resource, were critical to individual success.

The uncertainty created a need for the provision of other resources as well. Monitoring student progress during small group and whole group discussions allowed me to recognize the depth of student understanding as well as misunderstandings. The careful selection of questions and unremitting attention to student thinking helped me to provide resources that would make student access to the task possible. Questions ranged from those that might help students redirect themselves toward a more productive direction, to others that served to advance student thinking toward deeper understanding or a more sophisticated solution. At times the entire class needed a resource, such as the Core Tools demonstration described in the study. At other times, one student needed to be reminded of a prior task upon which he could draw. The provision of resources was apparent in whole group discussions of lessons as well. The strategic selection of student solutions, and the order in which they presented them, offered a resource to all of the students; encouraging discussion and helping students to make mathematical meaning (Stein, Engle, Smith, & Hughes, 2008). The resulting whole group discussion, offered students the opportunity to critique the reasoning of peers as they verbalized their own mathematical

understanding. They learned quickly that evidence was an essential element of the reasoned disagreements, and nearly every student became fluid at the public thinking process, prevalent in the class. The teacher was no longer the single authority; nor were students accountable to her alone.

In summary, the selection of the mathematical task was among the most critical decisions made with regard to enacting the principles of productive disciplinary engagement. Features of the task that have been highlighted as essential elements toward the creation of an environment wherein the principles of productive disciplinary engagement are apparent, are summarized below. In addition to engaging students at a high level cognitively, tasks should be chosen that have some or all of the following characteristics: 1) encourage students to take a position and justify it mathematically, 2) provide several solution paths that are not prescribed, 3) encourage students to sort and classify information, 4) offer several entry points depending on students' prior understanding and what they notice, and 5) solve using several representations. These characteristics were present in the tasks that were chosen for this study, and that contributed to the creation of an environment that supported the principles of productive disciplinary engagement. Not every task included all of these features, but most tasks included several of the aforementioned features.

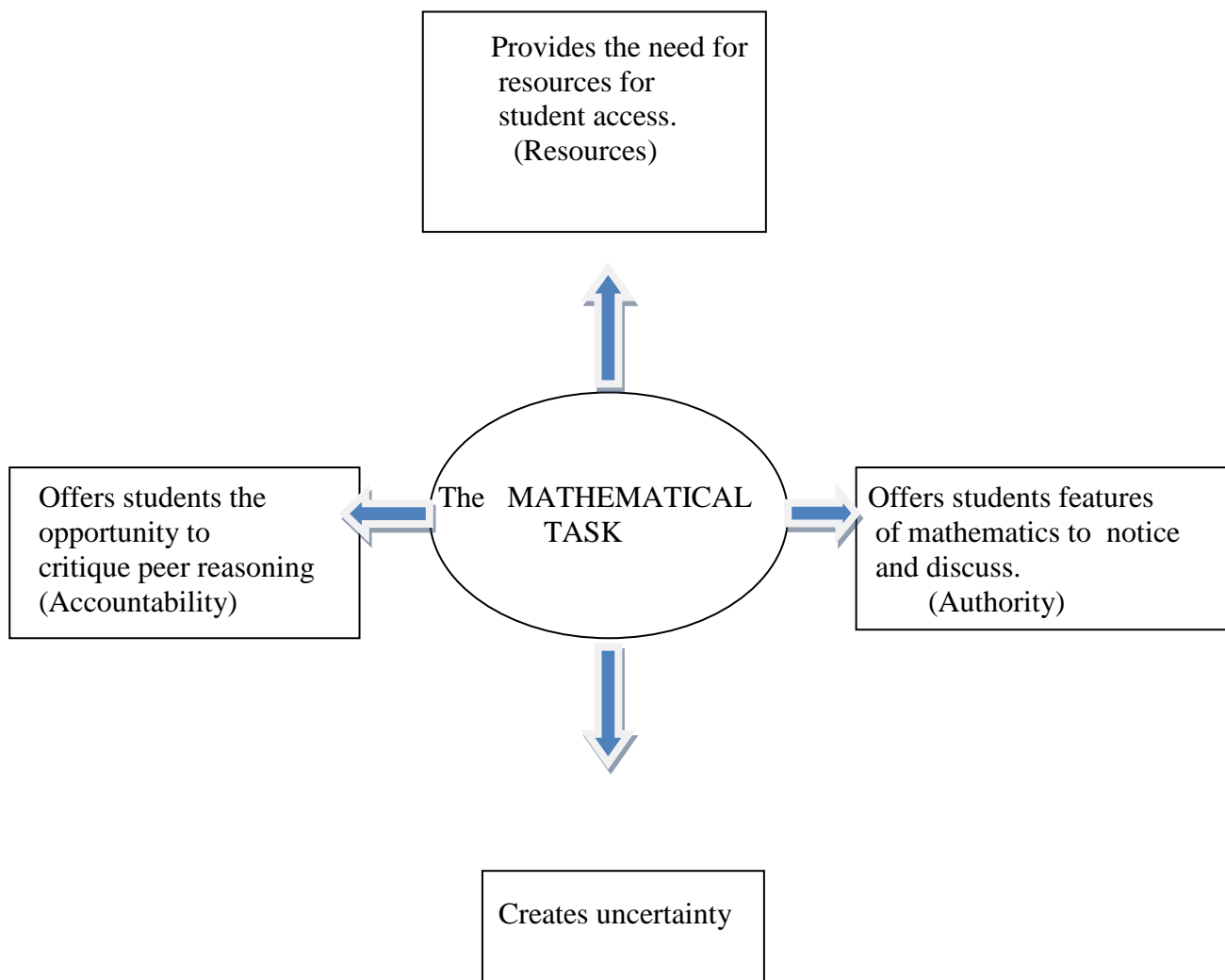


Figure 16. The Role of the Mathematical Task in the Implementation of the Principles of Productive Disciplinary Engagement

5.2.2 The principles of productive disciplinary engagement: A useful tool for practitioners

In this section, I address the potential value of this study for teachers and teacher educators. Specifically I consider the lessons from this study that might inform the practice of others. In addition, I discuss the adequacy of this framework for the design of a learning environment.

From the standpoint of a practitioner, the principles of productive disciplinary engagement are very useful as a design tool. Using the principles as a framework for lesson

planning focuses thinking related to many instructional practices that are consistent with the CCSS-M Mathematical Practices. For example, during lesson planning, with the four principles before me, I selected the task that students would address, considering those within and outside the selected curriculum. I considered the principle of problematizing, and what student uncertainty might be created through the introduction of the task. In addition, planning the enactment of the task included considerations related to the ways I would share authority with students. I mentally rehearsed the lesson, considering where I would stand and the best ways to allow students to assume roles of responsibility. Anticipating student solutions as part of the planning process (Stein, Engle, Smith, & Hughes, 2008) gave me confidence in offering students authority since I had already considered the most likely solutions to the tasks. In addition, by anticipating student solutions, I was able to preplan some of the questions I used to assess or advance student understanding. By preplanning the questions, I reduced the likelihood of asking a series of known-answer questions; a plan that the results indicate was successful. In general, the Five Practices (Stein, Engle, Smith, & Hughes, 2008) were very helpful to me in creating an environment that supported the principles of productive disciplinary engagement. .

Using the principles of productive disciplinary engagement in planning may help to create instructional habits. I had begun to use the four principles in my lesson planning at the start of the year and by the time this study commenced, their consideration was a habit of mind. The fact that there are only four principles, made it easy for me to mentally check for their inclusion prior to teaching a lesson. Further, it streamlined the planning process for me because some of the habits that I had created for myself no longer needed to be written in the plan. For example, I developed the habit of sitting with small groups as they worked, thus placing myself in a teacher-as-partner stance. It became routine to ask questions that probed student thinking

and pushed them to think more deeply. Problematizing through careful task selection was also a routine of planning.

In addition, , using the principles of productive disciplinary engagement offers teachers and teacher educators a way to establish “new moorings” for efficacy p. 396., J.P. Smith,1996)). With the principles of productive disciplinary engagement as a measuring tool, teachers would have a way to gauge their success at the end of a lesson or set of lessons. These principles offer teachers a framework with which they could use to evaluate themselves and a reflection tool that would encourage continued growth. If teachers developed an understanding of each of the principles of productive disciplinary engagement, they could begin to establish new meaning for effective teaching. The principles of productive disciplinary engagement could replace the view that effective teaching means providing explanations and procedures, and could serve to provide teachers with a meaningful tool for implementing the CCSS-M Mathematical Practices.

It will not be enough to for teachers to learn the instructional tools related to enacting the principles of productive disciplinary engagement described in this document. Learning to use Accountable Talk, the teacher-as-partner stance, or teaching students to hold one another accountable may not be enough. There are several teacher attitudes that will impact the implementation of the four principles. First, implementing these instructional practices demands flexibility on the part of the teacher. For example, although I know that I will include the Making Purple task this year when I teach the unit related to probability, I can’t say for certain that it will be used on the seventh, the eighth, or the ninth day of the unit. When I will choose to use it depends on the students and their thinking at the time. The common practice of teachers using guided notes and exact lesson plans from the year prior are indicative of a view that planning is a static process and that students learn the same material the same way at the same

time. Planning the implementation of tasks is useful from year to year, but the plan for the entire lesson is much less predictable. For example, I would certainly use my notes from the Making Purple task next year. However, the lesson may not flow in exactly the same way. Students may need more time, have disagreements that must be addressed, or require some additional resources beyond what I used this year. I argue that fully embracing the principles of productive disciplinary engagement demands that teachers view planning as a dynamic process, and allow for some flexibility so that teaching is responsive to students.

Second, implementation demands attention to student thinking with preplanned questions. If a teacher is to plan each lesson using the thinking of students at a moment in time, then questions need to be focused on students' mathematical thinking. One needs a very accurate indication of student thinking. Since every student constructs meaning in a potentially different way, one needs to have many students talk; not just one or two. Hence, the participation pattern that includes many students is critical.

In addition, commitment to the principles on the part of the teacher is critical. Unswerving teacher commitment was necessary for months preceding this study in order to encourage students to participate in a way that was consistent with the principles of productive disciplinary engagement. While students frequently complained that "you didn't teach me how to do this", and parents emailed that "I wasn't teaching", my practice and my expectations were consistent. As the months progressed from August to December, students learned to adapt to new expectations. I knew they would. I knew they could be decision-makers. I knew they could assume authority and hold each other accountable. I knew that given the right balance of resources and tasks that students would participate in the way I had planned. My only question was how long it would take. My belief in the students' capacity to participate was unshakable.

The students' capacity to redefine success in mathematics class was largely due to the consistent and relentless effort on my part to use the instructional practices that I described in this document. They learned to participate in the way that I had envisioned through "immersion", much like students learn a new language when they move to a foreign nation. They learn because it is necessary to learn. Students recognize that the natives are not going to change their tongue to meet their needs and that they must be the ones who adapt.

Implementing the principles of productive disciplinary engagement in the way that I have describe, challenges the use of short, daily math classes for secondary students. It causes me to wonder if longer classes might lead teachers to construct lessons that focus more on student mathematical thinking. I say this because it is often true that teachers believe that at the conclusion of a forty-two minute lesson, students should exit understanding the same mathematics in the same way. Learning is viewed and measured using a very short increment of time. Traditional lesson formats allow little time for exploring and talking about mathematics. Often, lessons begin with a review of homework, some time for direct instruction, followed by time for students to practice skills. Conversely, my commitment to the belief that learning takes place over time, contributed to students' capacity to participate in the way I had planned. The underlying belief that not everyone learns the same thing at the same time, allowed me to be content that not everyone exhibited the same competencies at the conclusion of a forty-two minute lesson. In my view, though, everyone would develop similar competencies over the course of the unit of study. Lesson structure was influenced by this belief. Lessons were continuous and not implemented as discrete lessons wherein every lesson was a small, separate topic. Influenced by my view of learning as a continuum and also my view of mathematics as a series of interrelated ideas, the lessons often flowed from one to another without a distinct ending

point at the conclusion of the forty-two minutes. There were times where the discussion continued from one class to another, such as days four and five, and days five and six; where the resources provided during one class were provided based on the discussion the preceding day, such as the Core Tools simulation; where student work on a task was continued from the end of one period to the start of the next class, such as day six and seven lessons; where the class opened with addressing misconceptions that had become apparent in the preceding class; such as days ten and eleven. I argue that this continuous view of learning and the related integration of each day's instruction into the next, contributed to the students' capacity to participate in the way that it is described in this document. I would not argue that the principles could not be implemented using another strategy: only that the continuity was helpful. This continuity might be implemented by teachers more easily if the class length were longer. Certainly a planning document that supports this continuum of learning would help teachers to embrace this view.

Largely due to this view of learning, the rhythm of each class was authentic. That is, not every class followed the same format. The lesson each day was structured to be responsive to the understanding presented by the students the preceding day, and move students toward the learning goal. The lack of a repetitive lesson structure contributed to the sense that students were doing real work that was not contrived to fit into a preplanned portion of the lesson that necessarily was a certain length of time. In reality, lessons were planned using the rather traditional lesson plan format, but the enactment of each lesson reflected the continuous view of learning that allowed for lessons to flow from one to another. The responsiveness of the lessons to the students' developing understanding was one way that authority was shared with the students. Although it was probably not apparent to the students, in essence the enactment of my lessons and student thinking were reflexively related. Lessons were not "delivered". Content

was not “covered”. Using my content knowledge, setting clear goals for student learning, selecting tasks that would move students closer to the intended goal, and listening carefully for ways that students understood the content, lessons were enacted with students. Although I chose tasks based on cognitive demand, I also considered tasks based on developing a continuing student trajectory toward the learning goal. Using carefully selected tasks that offered students something worthy of talk, students were doing mathematics, engaging in mathematical thinking together, and making their thinking public.

5.2.3 Limitations of the study

Classroom instruction is a complex array of people, talk, activity, and mathematical content that is interwoven in an intricate, and changing design. It is impossible to capture all of the nuances that together create a classroom environment. Ideally, a researcher can capture those areas of most interest using video and sound equipment that provide the opportunity for repeated review. However, even these tools allow for the capture of only a small part of the intended environment and largely ignore the thinking of the teacher as she makes a myriad of decisions throughout the enactment of the lesson. Limitations with regard to what is captured in a lesson are compounded by limitations in the analysis of the data, interpretation of the data, and conclusions drawn from the data. Every element of the study depends on the views of the researcher and the lens through which she interprets information.

I have repeatedly stated that this study addresses productive disciplinary engagement (Engle & Conant, 2002), however this work speaks only to the disciplinary and engagement portions of the term. Using Engle and Conant’s definition for productive that is, “to make intellectual progress” then the study has not addressed the extent to which the lessons were

productive. In order for productive to be addressed, the study would have needed to include some measure of change in student understanding over an increment of time such as would be the case if one uses pre-tests and post-tests. Thus, the productive portion of the term was beyond the scope of this work and would be an excellent topic of subsequent research.

The definition of problematizing utilized throughout this study reflects the definition provided in Engle (2011) which is a refinement in the definition put forth in her original work on the subject (Engle & Conant, 2002). In the original work, the authors present the definition broadly to include the idea that “teachers should encourage students’ questions, proposals, challenges, and other intellectual contributions rather than expecting that they should assimilate facts, procedures and other answers” (p.404). In other words, students should be provided the opportunity to define problems that elicit their curiosities. It is in Engle (2011) that problematizing is more carefully described to include, “any individual or collective action that encourages disciplinary uncertainties to be taken up by students” (p.6). In the original study, the controversy in which students initiated and engaged, , was indicative of problematizing. Using the Engle (2011) definition, this instance would also qualify as an incidence of problematizing. However, the more recent definition focuses attention more on problematizing as initiated by the teacher. The teacher chooses tasks with certain kinds of features and in so doing, the teacher creates uncertainty for students. My view of uncertainty and problematizing is limited to the research with which I am familiar and the fourteen years of classroom experience that has influenced this work. It is through a lens created by the two, that I have described the principle of problematizing and the relationship to uncertainty.

It is difficult to conclude the extent to which every student was served by the implementation of the principles of productive disciplinary engagement. Based on the data, it is

apparent that some students spoke more than others in whole group settings. I can confidently state that the four special education students were actively engaged and spoke frequently. A casual observer would not have been able to identify which students received special education services except for the fact that I read all materials to one student. In order to confidently draw conclusions related to the extent to which the implementation was effective for each student, one could count turns of talk for each student and compare them, recognizing that some students learn without making their thinking public. There was one student (David) who flatly refused to make his thinking public, either orally or in writing, and was not an identified Special Education student. An episode that highlights one of my attempts to encourage his participation was discussed in Chapter 4. I would be very interested to examine the data of his small group interaction and note whether his participation in both activity settings were similar or not. My anecdotal evidence suggests that he was active in small groups. In addition, it would be very interesting to collect assessment data that would suggest a relationships between the extent to which students orally participate and the extent to which they moved toward the learning goals. All students didn't participate in whole group discussions at the same level. It would be interesting to determine to what extent those who participated to a lesser degree met the intended learning goals. I wonder if they were not as robustly engaged in learning or were they just learning silently (Hatano & Inagaki, 1998).

Finally, a reader may wonder whether the findings in this document are domain specific. Because the study entailed only one content area, probability, one might wonder if there was something specific about the topic that made it a particular fit with the implementation of productive disciplinary engagement. My experience suggests that the results were not domain specific. Productive disciplinary engagement may be accomplished in a variety of domains. I

believe that productive disciplinary engagement depends more on the tasks chosen than the domain in which the task lies. It would be interesting to repeat this study using another domain. Conducting a similar study may provide insight regarding commonalities and differences in student behaviors as they engage in learning another content topic.

5.2.4 Directions for further research

This study provides one glimpse of a classroom in which the principles of productive disciplinary engagement were evident. The decomposition of some of the supporting teacher and student behaviors will hopefully provide information for studies of larger size that might further this work. The work described here has helped me to identify other potential studies that may contribute to a more comprehensive understanding of the elements that impact the enactment of the *principles of productive disciplinary engagement*.

Creating a learning environment that supports the students in productive disciplinary engagement is a practical challenge for teachers. Although the application of these design principles have been investigated in educational environments, including the one in this study, the articulation of the way to create the environment has not been articulated. This study helps to define the teacher and student behaviors that are evident once the environment is created, however, it doesn't address its creation at all. It leaves the reader wondering what happened from the first day of school until the study commenced. Did students come already knowing how to participate in the way that is described in this study? The way to develop the environment can be inferred, in part, from the study, but much more work is needed to decompose the teacher practices in a way that might allow teachers and teacher educators to apply these principles in a variety of educational settings with teachers of varying background and experience.

What students *noticed* with regard to mathematical features of tasks and solutions was quite varied. Whole group discussion emphasized the differences and encouraged students to consider what their peers had noticed in this study. Much of the research in the mathematical literature has addressed what teachers notice, but what students notice is also a critical factor. Studies that address pedagogical moves that might encourage students to “see” mathematical patterns, encourage student capacity for sorting information, and choose salient features of tasks would contribute to our understanding of the way students learn and the instruction that might support learning.

In addition, research related to what teachers need to know in order to enact the principles of productive disciplinary engagement will inform teacher education. For example, I wonder what contributes to a teacher being willing to battle external constraints that might limit the enactment of the principles of productive disciplinary engagement? What contributes to a teacher’s capacity to balance student frustration with their new authority to solve problems as they learn a new way to participate in mathematics class, and the teacher’s own feeling of success?

5.3 SUMMARY

This study examined the ways that the teacher and students enacted the *principles of productive disciplinary engagement* in one classroom. Further it considered the work of the teacher as she purposefully utilized the *principles of productive disciplinary engagement* in constructing an environment. The results provided insight related to the actions of the students and the teacher during a unit of study that encompassed fifteen days. Finally, although the study addresses the

research questions, it draws attention to other potential questions that may be answered in future studies.

APPENDIX A

STUDENT QUESTIONNAIRE

ME Williams' Dissertation Study

2013-2014

Date: _____

I want to know what you think!

This is NOT a test. There are NO wrong answers. I want to know what you think about math class this year.

Your answers are confidential. No one will be told what you answered. Your answers will be combined to reflect the attitudes and opinions of the entire class. This survey is anonymous.

This survey is voluntary. You do NOT have to answer any question that you do not wish to answer, but I hope you will answer as many questions as you can. Your answers will help me to become a better teacher and complete my school work.

This survey is not related to your grade. Your grade will not reflect participation or non-participation in the survey.

How much do you agree with the following statements about your teacher in your Math class: My teacher: (circle the answer that reflects your opinion)

1. Often connects what I am learning to life outside of the classroom.

Strongly disagree Disagree Agree Strongly agree

2. Encourages students to share their ideas about things we are studying in class.

Strongly disagree Disagree Agree Strongly agree

3. Often requires me to explain my answers.

Strongly disagree Disagree Agree Strongly agree

4. Encourages us to consider different solutions or points of view.

Strongly disagree Disagree Agree Strongly agree

5. Encourages students to stop working when the work gets hard.

Strongly disagree Disagree Agree Strongly agree

6. Expects us to work together to solve problems.

Strongly disagree Disagree Agree Strongly agree

7. Gives us work in class that is challenging.

Strongly disagree Disagree Agree Strongly agree

8. Wants us to become better thinkers, not just memorize things.

Strongly disagree Disagree Agree Strongly agree

9. Wants us to discuss possible solutions to problems with other students.

Strongly disagree Disagree Agree Strongly agree

10. Rarely asks students to show their work on the board or document projector

Strongly disagree Disagree Agree Strongly agree

11. Makes resources (graph paper, spinners, books) available to us in case we need it.

Strongly disagree Disagree Agree Strongly agree

How often does this happen in your mathematics lessons?

a) The teacher shows us how to do mathematics problems.

never once in a while usually always

b) We copy notes from the board.

never once in a while usually always

c) We work on mathematics tasks in small groups

never once in a while usually always

d) Students use the board or document projector

never once in a while usually always

e) The teacher uses the board or document projector

never once in a while usually always

f) If we don't know how to solve a difficult problem we ask other students for help.

never once in a while usually always

g) If we don't know how to solve a difficult problem we ask the teacher for help.

never once in a while usually always

h) Desks are organized in rows so that we can work independently.

never once in a while usually always

i) During student presentations, we ask questions of each other if we don't understand what he is explaining.

never once in a while usually always

j) I feel free to invent my own way to solve a mathematics problem.

never once in a while usually always

k) Students read and work from the textbook while the teacher talks about it.

never once in a while usually always

l) Students listen while the teacher explains rules and definitions.

never once in a while usually always

m) Students, together with the teacher, decide whether an answer is correct.

never once in a while usually always

APPENDIX B

QUESTIONING FRAMEWORK

Table 15. Question Framework

Question Type	Description	Example
Gathering information, leading students through a method	Requires immediate answer. Rehearses known facts/procedures. Enables students to state facts/procedures	What is the value of this equation? How would you plot this point?
Inserting terminology	Once ideas are under discussion, enables correct mathematical language to be used to talk about them	What is this called? How would you write this correctly?
Exploring mathematical meanings and/ or relationships	Points to underlying mathematical relationships and meanings. Makes links between mathematical ideas and representations	What is this x on the diagram? What does probability mean?
Probing, getting students to explain their thinking	Ask students to articulate, elaborate, or clarify ideas	How did you get 10? Can you explain your idea?
Generating Discussion	Solicits ideas from other members of the class	Is there another opinion about this? What did you say Justin?
Linking and applying	Points to relationships among mathematical ideas and mathematics and other areas/life	In what other situation could you apply this? Where else have we used this?
Extending thinking	Extends the situation under discussion to other situations where similar ideas may be used	Would this work for other numbers?
Orienting and focusing	Helps students focus on key elements or aspects of the situation in order to enable	What is the problem asking you? What is

Table 15 (continued)

	problem solving	important about this?
Establishing context	Talks about issues outside of math in order to enable links to be made with mathematics	What is the lottery? How old do you have to be to play the lottery?

APPENDIX C

THE MATHEMATICS TASK ANALYSIS GUIDE

Lower-Level Demands	Higher-Level Demands
<p style="text-align: center;"><u>Memorization Tasks</u></p> <p>Involves either producing previously learned facts, rules, formulae, or definitions OR committing facts, rules, formulae, or definitions to memory.</p> <p>Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.</p> <p>Are not ambiguous – such tasks involve exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated.</p> <p>Have no connection to the concepts or meaning that underlie the facts, rules, formulae, or definitions being learned or reproduced.</p>	<p style="text-align: center;"><u>Procedures With Connections Tasks</u></p> <p>Focus students’ attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.</p> <p>Suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.</p> <p>Usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning.</p> <p>Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.</p>
<p style="text-align: center;"><u>Procedures Without Connections Tasks</u></p> <ul style="list-style-type: none"> Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the 	<p style="text-align: center;"><u>Doing Mathematics Tasks</u></p> <p>Requires complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example).</p> <p>Requires students to explore and to understand the nature of mathematical concepts, processes, or relationships.</p>

<p>task.</p> <ul style="list-style-type: none"> • Require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. • Have no connection to the concepts or meaning that underlie the procedure being used. • Are focused on producing correct answers rather than developing mathematical understanding. • Require no explanations, or explanations that focus solely on describing the procedure that was used. 	<p>Demands self-monitoring or self-regulation of one's own cognitive processes.</p> <p>Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.</p> <p>Requires students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions.</p> <p>Requires considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.</p>
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Figure 17. The Mathematics Task Analysis Guide (Stein, Smith, Henningsen, & Silver, 2000)

APPENDIX D

TASK RUBRIC EXAMPLE

Summative Assessment- Probability

Question 1- Part A

3 A. The answer reflects an understanding that the probability that a player gets H/H or T/T results in her likely losing \$3. The explanation supports the correct answer with enough detail to explain not only what the outcomes might be but also the likelihood of each in writing that follows a logical sequence.

2-A. The answer reflects an understanding that the probability that a player gets H/H or T/T results in her likely losing \$3. However, the explanation is not detailed enough or does not follow a logical sequence that allows the reader to understand the student's reasoning process or the student has used faulty reasoning.

1-A. The answer is incorrect but the student has demonstrated some understanding of the likelihood of specific outcomes. The explanation reflects a partial understanding of the mathematical reasoning behind the answer.

0-A. The answer is incorrect with no demonstrated understanding of the likelihood of the specific outcomes. The explanation does not reflect a logical explanation using mathematical reasoning. Alternatively, the student has not answered the question.

APPENDIX E

LESSON PLAN FORMAT EXAMPLE

Table 16. Lesson Plan Format Example

Lesson Objective, Standards, and CCSS- Mathematical Practices	
<p>Make sense of problems and persevere in solving them.</p> <p>Model with mathematics</p> <p>Use tools strategically</p> <p>Construct viable arguments and critique the reasoning of others.</p> <p>Look for and make use of structure</p> <p>Look for and express regularity in repeated reasoning.</p>	<p>Use random sampling to draw inferences about a population.</p> <p>1. Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.</p> <p>2. Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. <i>For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.</i></p> <p>Draw informal comparative inferences about two populations.</p> <p>3. Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. <i>For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.</i></p> <p>4. Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. <i>For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.</i></p> <p>Investigate chance processes and develop, use, and evaluate probability models.</p>
	<p>5. Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around $\frac{1}{2}$ indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.</p> <p>6. Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. <i>For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.</i></p>

Table 16 (continued)

	<p>7. Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.</p> <p>a. Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. <i>For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.</i></p> <p>b. Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. <i>For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?</i></p> <p>8. Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.</p> <p>a. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.</p> <p>b. Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., “rolling double sixes”), identify the outcomes in the sample space which compose the event.</p> <p>c. Design and use a simulation to generate frequencies for compound events. For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?</p>
<p>Procedure</p> <p>Resources; area model that we constructed for Making Purple, last week.</p>	<p>Probability- Day 12</p> <p>Focus question: How can you use experimental or theoretical probabilities of a compound event to predict the number of times one particular combination will occur out of any given number of repetitions of the event?</p> <p>Review how to analyze a two-stage outcome using an area model. Have students turn to pg. 75 in CMP. Review our discussion of last week. Are purple and not purple equally likely? How might you figure it out exactly? For Spinner A, what is the likelihood of getting red? How is this represented on the square? Distribute the area model that we agreed to for that scenario.</p> <p>Address question D. Let students work in pairs to complete. Every person must turn in a written response. Following student completion, engage students in a discussion of their thinking. If time permits, begin the next lesson: regarding one and one free throws.</p> <p>Homework #8, page 82. Complete written explanation is expected.</p>
Assessing and advancing questions	<p>What does each partition in the side of the square represent? What does the area of each section represent? How many regions are there? Why? What do you notice about the sections when there is an equal number of each color? When it is different? What other numbers of each color might this same table represent?</p>
Ways that authority will be shared with students	<p>Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. I am removing some authority today by insisting on one representation, but it is necessary.</p>
Ways that students will be held accountable to each other and the teacher	<p>Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.</p>

Table 16 (continued)

<p>Student resources that will be made available</p>	<p>Red/blue chips, graph paper, peers, .</p>
<p>Problem solving: Ways that students will be challenged in ways that engender genuine uncertainty</p>	<p>Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution.</p> <p>My intent is that through questioning I can help students to make connections between probability and algebraic reasoning.</p>

APPENDIX F

ACCOUNTABLE TALK MOVES

Table 17. Accountable Talk Moves

Talk Move	Function	Example
Marking	Direct attention to the value and importance of a student's contribution	That's an important point.
Challenging	Redirect a question back to the students, or use students' contributions as a source for further challenge or query	Let me challenge you: Is that always true?
Modeling	Make one's thinking public and demonstrate expert forms of reasoning through talk.	Show us your thinking. Here's how a mathematician works.
Recapping	Make public in a concise, coherent form, the group's achievement at creating a shared understanding of the phenomenon under discussion	Let me put these ideas together. What have we discovered?

Table 17. (continued)

To Support Accountability to Community

Keeping the Channels Open	Ensure that students can hear each other, and remind them that they must hear what others have said	Sya that again and louder. Can someone repeat what was just said?
Keeping Everyone Together	Ensure that everyone not only heard, but also understood what the speaker said	Can someone add on to what was just said? Did everyone hear that?
Linking Contributions	Make explicit the relationship between a new contribution and what has gone before	Does anyone have a similar idea? Do you agree or disagree with what was said? Your idea sounds similar to his idea.
Verifying and Clarifying	Revoice a student's contribution, thereby helping both speakers and listeners to engage more profitably in the conversation	So are you saying...? Can you say more? Who understands what was said?

To Support Accountability to Knowledge

Pressing for Accuracy	Hold students accountable for the accuracy, credibility, and clarity of their contributions	Why does this happen? Someone give me the term for that.
Building on Prior Knowledge	Tie a current contribution back to knowledge accumulated by the class at a previous time.	What have we learned in the past that links with this?

To Support Accountability to Rigorous Thinking

Pressing for Reasoning	Elicit evidence to establish what contribution a student's utterance is intended to make within the group's larger enterprise	Say why this works. What does this mean? Who can make a claim then tell us what their claim means?
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Table 17. (continued)

Expanding Reasoning	Open up extra time and space in the conversation for student reasoning	Does the idea work if I change the context? Use bigger numbers?
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(O'Connor & Michaels, 1993)

APPENDIX G

CODING SCHEME FOR THE PRINCIPLES OF PRODUCTIVE

DISCIPLINARY ENGAGEMENT

Table 18. Disciplinary Engagement

Code	Definition		Principle
T	Teacher		
S	Student		
U	Uncertainty expressed orally	“ I don’t get it.”; “ What do you mean?”	Problematizing
US	Student uncertainty	“ I don’t understand.”	Problematizing
UT	Teacher uncertainty	“ I don’t understand what you mean.”	Problematizing
QSS	Student to student question (not asking for a justification)	“Does this remind you of when we studied proportions? “	Problematizing
QTS	Teacher question or comment to student or class that highlights differences in student conjectures or asks for a conjecture or asking a question that requires students to generalize or extend their thinking	“Does anyone have another opinion?” or “ Do you agree with Jack’s answer or John’s answer?” or “It works in example x, will it work in example y?” “Can you generalize	Problematizing/ Accountability
JS	Student justifying reasoning without prompt	“I know the answer is 2 possible outcomes because	Accountability

Table 18 (continued)

JWT	Justifying reasoning as a result of a prompt by the teacher	Teacher:” Can you say more about that?” Student: “I know it is two because of the distributive property. “	Accountability
JWS	Justifying reasoning as a result of a prompt by a student	Student 1:” I didn’t get what you got. Can you explain your answer?” Student 2 : “The way I was thinking about this was....”	Accountability
ATT	Teacher holds students accountable using Accountable Talk or the insertion of mathematical vocabulary.	See Appendix F	Accountability
ATS	Accountable talk used by the students	See Appendix F	Accountability
T or S	Teacher or student talk turn		Authority
AT	Teacher provides answer or assumes authority	“The answer to #1 is 7”.	Authority
AS	Student publicly provides answer (correct or incorrect); makes conjecture	“I got 3 as the most likely outcome”.) (This statement gives classmates the opportunity to critique, placing them in an authority position.)	Authority
ASC	Student self corrects previously flawed or incomplete explanation or answer.	“ I see now. It’s not $\frac{1}{2}$, it’s more like $\frac{2}{3}$ because of the number of pieces.”	Authority
N	Noticing. Students select information from competing information. They are challenged to sort through visual cues, in an effort to identify particular mathematical features among competing bits of mathematical information.	“Although the numbers are larger they are the same proportion. The part to whole relationship is the same.”	Authority
PTP	Positioning- teacher positions herself as a peer	“Let’s try to figure this out.” “How are we doing on this task?”	Authority
PTA	Positioning- teacher positions herself as an authority	“This isn’t right. The way to do this is to use a tree diagram.”	Authority

Table 18 (continued)

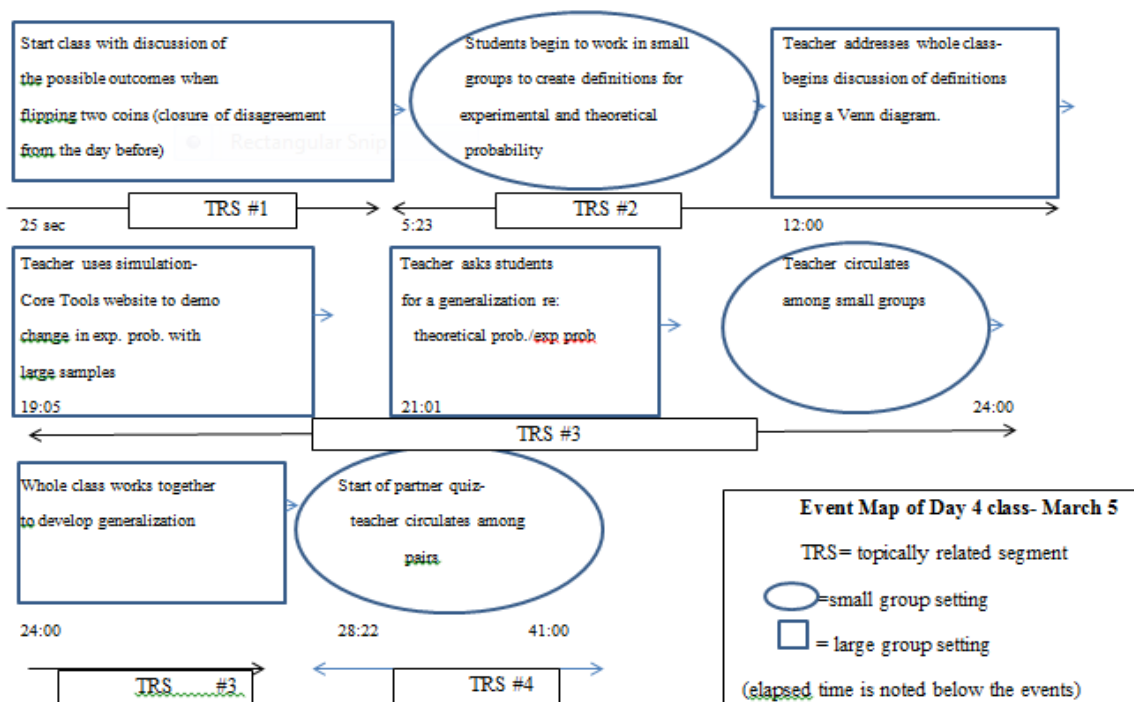
PTS	Positioning- teacher positions student in authority position-capable independent decision-makers	<p>“MaLyla’s solution is a good example of a tree diagram.” (Draws attention to her as the author.)</p> <p>“ I notice that you chose to use an area model. I am wondering why you made that choice?” (This question begs for an explanation regarding an authorial decision; requiring an explanation of something that may have been an unconscious decision. The student must don an author’s hat to provide the narrative.</p> <p>Do you agree with what Lynsey just said? (This begs an authorial decision: requiring an explanation in the form of a narrative. It also assumes that the student is capable of an independent decision.)</p> <p>“Thanks for straightening me out.” This comment implies that the student has helped the teacher understand more fully. The teacher is fallible and is engaged in the same intellectual work. Correcting errors is a joint concern.</p>	Authority
PSS	Positioning- student places another student in an authority position	Lynsey’s way works really well for me. (Student places Lynsey in the position of an author)	Authority

Table 18 (continued)

PS	Positioning- student exhibits authority through the act of publically disagreeing or agreeing with a peer or the teacher, by answering a peer question, or by walking to the overhead or another group to make a point.	"I don't think I agree with her logic."	Authority
RT	Resource provided by the teacher- may be ideational, relational, or material or more than one; or she states that she expects students to use a resource.	Ideational- "Remember the problem we did with unit rates?" Material- coins, counters, graphs, Relational- peers- may include directions regarding the way to participate. "I want one explanation for your whole group".	Resource
RS	Resource as above, provided by a peer. Students are providing resources to each other or are using each other as resources ; such as listening to an explanation then adding to it	"Like Bob said, it makes more sense to look at the area of each segment of the area model."	Resource

APPENDIX H

EVENT MAP EXAMPLE



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Figure 18. Event Map Example

APPENDIX I

TASKS USED IN THE STUDY

Day 1

Problem 1.1 (Lappan et.al, 2014, p. 9)

Problem 1.1

✓ **A** 1. Test your prediction. Toss a coin 30 times (one for each day in June). Record your results in a table with the headings below. Your table will have 30 rows.

Coin-Toss Results

Day	Result of Toss (H or T)	Number of Heads So Far	Fraction of Heads So Far	Percent of Heads So Far
1	■	■	■	■
2	■	■	■	■

2. As you add more data, what happens to the percent of tosses that are heads?

✓ **B** Work with your classmates to combine the results from all the groups.

1. What percent of the total number of tosses for your class is heads?
2. As your class adds more data, what happens to the percent of tosses that are heads?
3. Based on what you found for June, how many times do you expect Calvin to eat Cocoa Blast in July? Explain your reasoning.

✓ **C** Calvin's mother tells him that the chance of a coin showing heads when he tosses it is $\frac{1}{2}$. Does this mean that every time he tosses a coin twice, he will get one head and one tail? Explain.

A C E Homework starts on page 17.

Figure 19. Problem 1.1

Orally assigned task; Graph the percent of heads versus the number of tosses.

They were asked to theorize regarding the shape of the graph when the number of tosses increased to very large numbers. (teacher developed)

Day 2

Problem 1.2 (Lappan, et.al., 2014)- part A, B

Part A: Conduct an experiment to test your prediction about how a paper cup lands.

Toss a paper cup 50 times. Make a table to record your data.

Part B: Use your results to answer the following questions:

1. For what fraction of your 50 tosses did the cup land on one of its ends? What percent is this?
2. For what fraction of your 50 tosses did the cup land on its side? What percent is this?
3. Do the landing positions end and side have the same chance of occurring? If not, which is more likely? Explain.

Define experimental versus theoretical probability in small groups (teacher developed)

Homework: problems 1-5 on page 17

1. Mikki tosses a coin 50 times, and the coin shows heads 28 times. What fraction of the 50 tosses is heads? What percent is this?
2. Suppose Calvin tosses a coin to determine his breakfast cereal every day. He starts on his 12th birthday and continues until his eighteenth birthday. About how many times would you expect him to eat Cocoa Blast cereal?
3. Calvin tosses a coin five days in a row and gets tails every time. Do you think there is something wrong with the coin? How can you find out?
4. Len tosses a coin three times. The coin shows heads every time. What are the chances the coin shows tails on the next toss? Explain.
5. Is it possible to toss a coin 20 times and have it land heads up 20 times? Is this likely to happen? Explain.

Day 3

Problem 6A, page 17. (Lappan, et.al., 2014)

Calvin tosses a paper cup once per day each day for a year to determine his breakfast cereal. Use your results from Problem 1.2 to answer the following.

- a. How many times do you expect the cup to land on its side? On one of its ends?
 - b. How many times do you expect Calvin to eat Cocoa Blast in a month? In a year?
- Explain.

Problem 1.3, A, B, C (Lappan, et.al., 2014)

Part A:

1. Conduct an experiment by tossing a pair of coins 30 times. Keep track of the number of times the coins match and the number of times no match occurs.

2. Based on your data, what is the experimental probability of getting a match?

Getting a no-match?

Part B:

Combine your data with your classmates' data.

1. Find the experimental probabilities for the combined data. Compare these probabilities with the probabilities in Question A.

2. Based on the class data, do you think a match and a no-match have the same chance of occurring? Explain.

Day 4

Partner Quiz: Task 1, Task 2, Task 3A, Task 3B

1. The probability of a particular event is $\frac{3}{8}$. What is the probability that the event will not happen? Explain.

2. Multiple choice. Which of the following numbers could not be a probability? Explain.

A. $\frac{1}{3}$ B. 0 C. $\frac{8}{9}$ D. 1 E. $\frac{5}{4}$

3. Juanita is holding 5 coins with a total value of 27 cents.

a. What is the probability that three of the coins are pennies? Explain your reasoning.

b. What is the probability that one of the coins is a quarter? Explain your reasoning.

Homework: #19, page 20 (Lappan, et.al., 2014)

19. Colby rolls a number cube 50 times. She records the result of each roll and organizes her data in the Table 4.3 below.

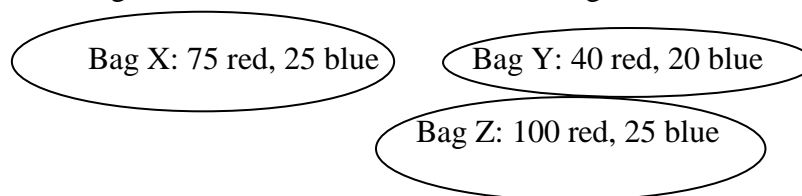
Number Cube Results	
Number	Frequency
1	
2	
3	
4	
5	
6	

- What fraction of the rolls are 2's? What percent is this?
- What fraction of the rolls are odd numbers? What percent is this?
- What percent of the rolls is greater than 3?
- Suppose Colby rolls the number cube 100 times. About how many times can she expect to roll a 2? Explain.

Day 5

Marble task (Silver, et.al, 1995)

Mrs. Rhee's math class was studying statistics. She brought in three bags containing red and blue marbles. The three bags were labeled as shown below.



Mrs. Rhee shook each bag. She asked the class, “ If you close your eyes, reach into the bag, and remove 1 marble, which bag would give you the best chance of picking a blue marble?

Which bag would you choose? ____

Explain why this bag gives you the best chance of picking a blue marble.

You may use the diagrams above in your explanation .

Day 6

No new tasks

Day 7

Problem 2.3, A2 presented orally

A.2. How many possible outcomes are there when you toss three coins?

Are the outcomes equally likely?

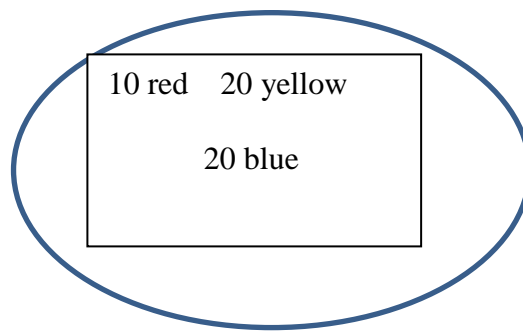
Cafeteria problem (Lappan, et.al, 2014)

(Provided in writing) Today, the school's cafeteria is offering a choice of pizza or spaghetti. You can get milk or juice to drink. For dessert you can get pudding or an apple. You must take one of each choice. Draw a tree diagram to show all the possibilities.

Day 8

Marbles task

The bowl below contains the indicated number of marbles. The marbles are well-mixed in this bowl. Juan believes that his chance of picking a blue marble is the same as his chance of picking a yellow marble. Is Juan correct? Explain your answer. (NAEP released item)

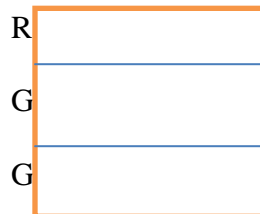


Sticky Gum Problem (Silver, et.al, 1995)

A penny bubble gum machine is filled with red and white gumballs. Mrs. Jones' twins want to have the same color gumball. How many pennies must Mrs. Jones be prepared to spend to be sure she gets a pair of matching gumballs? Now suppose Mrs. Jones' has triplets. How many pennies must Mrs. Jones be prepared to spend to be sure she gets three matching gumballs?

Day 9

Problem 4.1 A and 4.1 C (Lappan, et.al, 2014)



Miguel adds to his diagram to help him find the theoretical probabilities of drawing marbles from Bucket 1.

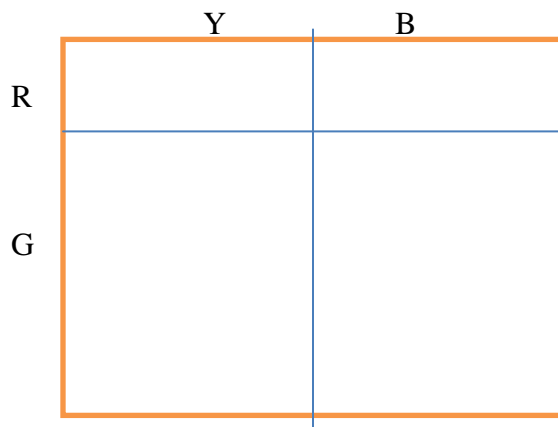
Part A

1. Explain what Miguel has done so far. Does this look reasonable?
2. Use the top edge to represent Bucket 2. How many sections do you need to represent the marbles in Bucket 2? Draw the lines and label the sections you need to represent Bucket 2.
3. Now label each of the sections inside the square with two letters to represent the results of choosing two marbles. RR in a section would mean that two red marbles were drawn from the buckets.

Part C

The area model below represents a different situation from Questions A and B. In this area model, $P(RY)=1/10$,

$P(RB)=1/10$, $P(GY)=4/10$, and $P(GB)=4/10$. Use the area model and these probabilities to answer the following questions:



4. Which of the following could be the contents of the two buckets?

Explain your reasoning.

- a. 2 red and 8 green in bucket 1; 5 yellow and 5 blue in bucket 2
- b. 2 red and 8 green in bucket 1; 10 yellow and 10 blue in bucket 2
- c. 1 red and 4 green in bucket 1; 3 yellow and 3 blue in bucket 2

ACE

Applications | Connections | Extensions

Applications

A school carnival committee features a different version of the Making Purple game, as shown below.

1. Before playing the game, do you predict that the school will make money on this game? Explain.
2. Use an area model to show the possible outcomes for this game. Explain how your area model shows all the possible outcomes.
3. What is the theoretical probability of choosing a red and a blue marble on one turn?
4. Suppose one marble is chosen from each bucket. Find the probability of each situation.
 - a. You choose a green marble from Bucket 1 and a yellow marble from Bucket 2.
 - b. You do not choose a blue marble from either bucket.
 - c. You choose two blue marbles.
 - d. You choose at least one blue marble.

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What Do You Expect?

Figure 20. Problems 1-4

Day 11

Each of the 6 faces of a fair cube is painted red, yellow, or blue. This cube is rolled 500 times. The Table 4.3 below shows the number of times each color landed face up. Based on these results, what is the most likely number of yellow faces on the cube. (NAEP)

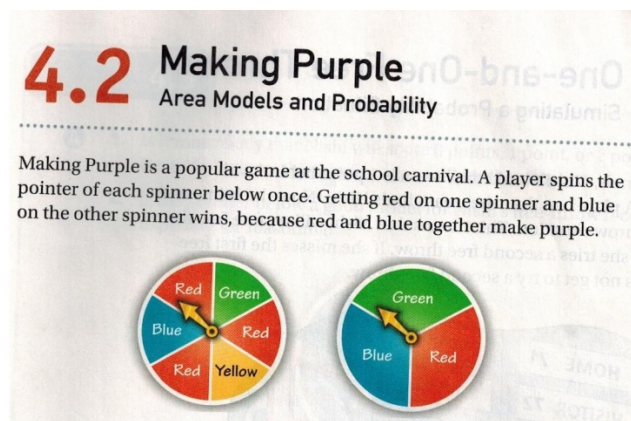
Red 100

Yellow 340

Blue 60

Day 12

Problem 4.2 Part D (Lappan, et.al., 2014)



Part D: The cost to play the game is \$2. The winner gets \$6 for making purple. Suppose 36 people play the game.

1. How much money will the school take in from this game?
2. How many people do you expect to win a prize?
3. How much money do you expect the school to pay out in prizes?
4. How much profit do you expect the school to make from this game?
5. Should the school include this game in the carnival? Justify your answer using your answers from parts 1-4.

Problem 4.3 Part B revised- provided partially completed area model

(Lappan, et.al., 2014)

In the district finals, Nishi's basketball team is 1 point behind with 2 seconds left. A player on the other team fouls Nishi. Now she is in a one-and-one free-throw situation. This means that Nishi will try one free throw. If she makes it, she tries a second free throw. If she misses the first free throw, she does not get to try a second free throw.

An area model representing the possible outcomes for Nishi's one-and-one free throws is shown below. Explain what you know about the design of the area model (This task was adapted to include a partially completed area model as below)

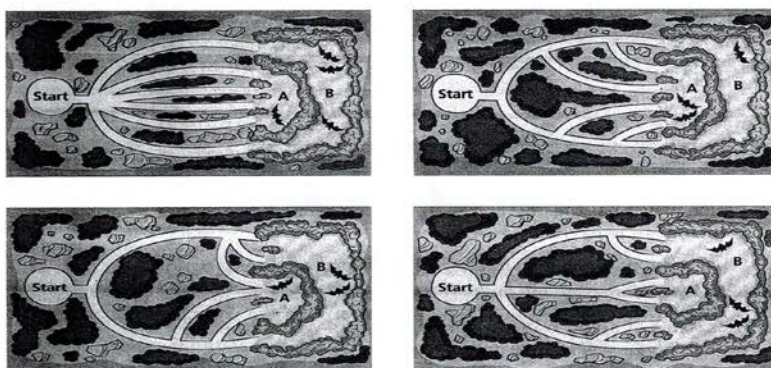
1. Why are the blocks the size they are?
2. How would you label the left side of the area model? Explain.
3. How would you label the top side of the area model? Explain.



Day 14 - (orally presented) How might you simulate the one-and-one free throw problem we did yesterday, using dice or a spinner? (adapted from Lappan, et.al, 2014, Problem 4.3A.3, p.77)

Day 15- Problem 15- Cave paths task (Lappan,et.al., 2014)

15. Multiple Choice Choose the map that the area model in Exercise 14 could represent.



Choice 1

B	A	A
A	A	B

Day 15 Assessment item

Name: _____ Class: _____ Date: _____

ID: A

probability summative

Short Answer

1. To play the Nickel Game, a player tosses two nickels at the same time. If both nickels land tails up, the player wins \$1. If both nickels land heads up, the player wins \$2. Otherwise, the player wins nothing.
 - a. If it costs \$1 to play the Nickel Game, how much could a player expect to win or lose if he or she plays the game 12 times? Explain your reasoning.
 - b. At next year's carnival, the game committee wants to charge prices that will allow players to break even. How much should they charge to play the Nickel Game? Explain your reasoning.
2. Mandy has a bag containing one green block (G), one brown block (B), and one yellow block (Y). She conducted 20 trials in which she drew one block from the bag and then flipped a fair coin. Here are the results of her experiment:

Color	G	Y	G	G	Y	Y	B	Y	G	Y	B	Y	B	Y	Y	B	G	G	Y	Y
Coin	T	T	H	T	H	T	T	H	T	T	T	H	T	T	T	T	H	T	H	T

- a. What is the experimental probability of drawing the brown block and flipping heads? What is the theoretical probability?
 - b. What is the experimental probability of drawing the yellow block and flipping tails? What is the theoretical probability?
 - c. How would you explain the differences you found between the experimental and theoretical probabilities?
3. Ann Marie has a spinner that is divided into four regions. She spins the spinner several times and records the results in a table. Based on her results, make a drawing of what the spinner might look like.

Region	1	2	3	4
Number of Times Spinner Lands in That Region	9	4	12	11

APPENDIX J

LESSON PLAN AND REFLECTION

(DAY 1)

Table 19. Lesson Plan and Reflection(Day 1)

Lesson Objective, Standards, and CCSS- Mathematical Practices	
<p>Make sense of problems and persevere in solving them.</p> <p>Model with mathematics</p> <p>Use tools strategically</p> <p>Construct viable arguments and critique the reasoning of others.</p>	<p>Use random sampling to draw inferences about a population.</p> <p>1. Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.</p> <p>2. Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. <i>For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.</i></p> <p>Draw informal comparative inferences about two populations.</p> <p>3. Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. <i>For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.</i></p> <p>4. Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. <i>For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.</i></p> <p>Investigate chance processes and develop, use, and evaluate probability models.</p> <p>5. Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.</p>

Table 19 (continued)

	<p>6. Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. <i>For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.</i></p> <p>7. Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.</p> <p>a. Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. <i>For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.</i></p> <p>b. Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. <i>For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?</i></p> <p>8. Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.</p> <p>a. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.</p> <p>b. Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., “rolling double sixes”), identify the outcomes in the sample space which compose the event.</p> <p>c. Design and use a simulation to generate frequencies for compound events. <i>For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?</i></p>
Procedure	<p>Probability- Day 1</p> <p>Give students time to review the intro individually, ask students to describe to their group some examples of probability situations, have a student summarize the scenario. Make a prediction in groups- groups test their prediction-Combine results of all groups</p>
Assessing and advancing questions	<p>What is the trend of the results? If you graphed the percent of the number of heads that were tossed, what would the graph look like? Calvin’s mother tells him that the chance of tossing a head is $\frac{1}{2}$. Does that mean that for every 2 tosses he will get one heads and one tails?</p>
Ways that authority will be shared with students	<p>Students must take charge and figure out what they don’t know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence</p>
Ways that students will be held accountable to each other and the teacher	<p>Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.</p> <p>Exit slip- What was the mathematical message of this lesson?</p>
Student resources that will be made available	<p>Peers. Pennies. Cups. Question examples hanging on the wall.</p>

Table 19 (continued)

Problematising: Ways that students will be challenged in ways that engender genuine uncertainty	<p>Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution.</p> <p>My intent is that through questioning I can help students to make connections between probability and proportional reasoning.</p>
---	---

2/28 Day 1- I am surprised that students evidently don't understand that more coin tosses is a better prediction than 30 tosses. Ed noticed that the percentage should hover around 50%, but at first many didn't even notice that. No student's graph represented the change in percentage of heads (as the number of flips increased) as moving closer to 50%. Students are talking freely, but still some are quiet. Students in Bob's group had a hard time getting a word in. Groups were very aware of the technical bias of flipping styles. Students still want to graph specific data—trends are not easy for them to envision. Most students were eager to engage and shared the flipping responsibility. Bryce, although not always engaged physically, seemed to notice the connection to proportional reasoning. I chose not to follow up on his comment with a class question because I didn't want to side track our discussion. I'm hoping that more students see the connection as time goes on in the unit.

Accountability- to peers in group discussion- to me through AT-

Authority- used board to record- shared ideas with small and large group- asked questions, critiqued the thinking of peers' graphs- students selected their roles

Resources- coin, data sheet, peers, graphs

Problematising- uncertainty as to what to conclude – creating the graph caused uncertainty in students, made them extrapolate their data in an abstract way.

APPENDIX K

LESSON PLAN AND REFLECTION

(DAY2)

Table 20. Lesson Plan and Reflection (Day 2)

	<p align="center">Probability- Day 2- cmp lesson 1.2 and 1.3</p> <p>Review homework: Students will work in groups for a short time to compare answers. I will circulate and ask questions especially focused on #4 that addresses independent events., also ensure that all have completed it (3 pts for completion) .</p> <p>Whole group discussion on the problem of their choice. Focus question: How does modeling with an experiment help you determine possible outcomes and the likelihood of each outcome? Connect with last lesson. Post the correct representation of the graph that students developed last class couched as another student's response. Whole group discussion. I am interested in the students seeing the merits of this graph themselves. So, why do you think the jagged parts are bigger at the start? What do you think of this line eventually comes down to 50%?</p> <p>Students will experiment with paper cups: their first experiment with events that are not equally likely. Will a cup behave like a coin and land on an end or side an equal number of times? Distribute paper so that students can develop their own recording scheme. Give individual time to consider Calvin's question on page 10. Students will be tossing cups 50 times. What data is worthy of collecting? Ask questions regarding whether the fractions are ratios. Use the words "experimental probability, theoretical probability, bias" with purpose. If time permits, begin lesson 1.3 (flipping two coins). Ask students to make a prediction about what is more likely to occur; a match or no match. Have groups record their data and discuss. Is a match/no match equally likely events? Why?</p> <p>Close: return to the focus question- How did we use modeling to develop experimental probability?</p> <p>Homework #6,7 on page 17.</p>
Procedure	
1:40 exit slip	

Table 20 (continued)

Assessing and advancing questions	What is the trend of the results? How do we develop the probability for the cup landing on a side or end? Is that a ratio? Connect to orange juice problem?
Ways that authority will be shared with students	Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. They have authority to construct their own record keeping system.
Ways that students will be held accountable to each other and the teacher	Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding. Exit slip- What was the mathematical message of this lesson?
Student resources that will be made available	Peers. Pennies. Cups. Question examples hanging on the wall.
Problematising: Ways that students will be challenged in ways that engender genuine uncertainty	Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution. My intent is that through questioning I can help students to make connections between probability and proportional reasoning.

3/3- Day 2 Students addressed the differences with cup vs coin flipping today. Talk was robust today; students easily talking in whole group. The timing of the lesson was tricky because while some groups were finished, others were still working. SOME students definitely noticed the value of a large sample size via the provided "student" graph while others are still working on it. Most students saw #4 of the homework as an independent event, although not all did. Henriette definitely saw them initially as dependent events. I notice that students are beginning to use the word "outcome" and several saw the relationship with proportional reasoning.

Accountability- students were accountable to me and peers for homework

Authority- used the Elmo to record their data, critiqued the reasoning of peers in a respectful way, students selected roles

Resources- cups, paper, graph of other "student"- ideas of peers

Problematising- interpreting the provided graph caused uncertainty related to the way to interpret it.

APPENDIX L

Lesson Plan and Reflection

(DAY 3)

Table 21. Lesson Plan and Reflection (Day 3)

Procedure	<p align="center">Probability- Day 3- cmp lesson 1.3 and 1.4</p> <p>Review homework: Students will work in groups for a short time to compare answers. I will circulate and ask questions especially focused on 6a that points to proportional reasoning(3 pts for completion) . Whole group discussion on the problem of their choice.</p> <p>Focus question: How do you determine the relative frequency of an outcome?</p> <p>Lesson 1.3 asks students to flip two coins. Students will be flipping two coins and determining the likelihood of a match or no match. They will first make a <u>prediction</u>, then flip the coins fifty times.</p> <p>So what did you find out about the probability of a match or no match?. Probe student inking to determine if they have considered all the possible outcomes T/T, H/H, T/H, H/T.</p> <p>Focus question:What does it mean for two events to be equally likely? Connect with last lesson. Was it equally likely for coins to match or not match?</p> <p>Give students quiet time to read Calvin's story about the penny on the train track.</p> <p>Why is Calvin's mother suspicious of the coin? What does it mean for a coin to be "fair? Read the story about names in a hat. Why is each card equally likely to be chosen but not each name? How many more of each name to add to make it equally likely? Have students address Problem 1.4 on page 16- discuss in small groups.</p> <p align="center">Homework #19 on page 20.</p>
1:35 pair exercise	<p>Close: Pair exercise- # 10 on page 18 of CMP- students can work in pairs or triples to complete.</p>
Assessing and advancing questions	<p>What are the possible outcomes? How did doing the experiment help you to visualize the outcomes? What does it mean to model mathematics? In what ways has modeling helped you? We have talked about visualizing many times, is there a connection?</p>
Ways that authority will be shared with students	<p>Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. They have authority to construct their own record keeping system.</p>
Ways that students will be held accountable to each other and the teacher	<p>Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.</p> <p>Exit slip- What was the mathematical message of this lesson?</p>

Table 21 (continued)

Student resources that will be made available	Peers. Pennies. Cups. Question examples hanging on the wall.
Problem-tizing: Ways that students will be challenged in ways that engender genuine uncertainty	<p>Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution.</p> <p>My intent is that through questioning I can help students to make connections between probability and proportional reasoning.</p>

3-4-14- Day 3 We began with a discussion of problem 6 that asked for students to scale yesterday's results regarding the probability of a cup landing on its side to establish the number of times it would land on its side in a year (365 days). Bryce quickly began to think proportionally, but was struggling with both the format for writing it and the language in helping others to understand his thinking. As he spoke at the document projector, I wrote a proportion to help students recognize the connection to proportional reasoning. I am disappointed that more students didn't quickly use proportional reasoning. Like my prior classes today, I don't think students considered this very extensively last night, although it was assigned. 19 d and e of the homework assignment also focuses on proportional reasoning. I will grade this assignment tomorrow. I don't usually grade assignments for correctness, only an attempt at completion- feel like it an equity issue. I will let them collaborate a few minutes before collecting the assignment. I am feeling very successful at encouraging students to express their thinking and feeling free to disagree. I noticed that they are really trying to provide evidence of their thinking, even without prompting. There are a few students still not talking much- Ed, Henry, Dennis. Dennis rarely does home work and when he does, I think someone else does it. Today he had something written, but couldn't talk about his work with group mates for problem 6A at the start of class. Most of his group didn't even attempt the homework. I will continue to work to get everyone talking. Students flipped two coins today. Nya and Estelle disagreed about what the potential outcomes might be and how many there were. Other students took sides. Discussion proceeded with very little intervention from me, including work on the white board. I asked them to consider their positions a little more for homework. I will encourage the group to draw a conclusion tomorrow regarding the theoretical probability of predicting a match.

APPENDIX M

LESSON PLAN AND REFLECTION

(DAY 4)

Table 22. Lesson Plan and Reflection (Day 4)

	<p>Probability- Day 4- cmp lesson 1.3 and 1.4</p> <p>Review homework: Students will work in groups for a short time to compare answers. I will circulate and ask questions especially focused on 6a that points to proportional reasoning(3 pts for completion) . Whole group discussion on the problem of their choice.</p> <p>Focus question: How do you determine the relative frequency of an outcome?</p> <p>Lesson 1.3 asks students to flip two coins. Students will be flipping two coins and determining the likelihood of a match or no match. They will first make a <u>prediction</u>, then flip the coins fifty times.</p> <p>So what did you find out about the probability of a match or no match?.</p> <p>Probe student thinking to determine if they have considered all the possible outcomes T/T, H/H, T/H, H/T.</p> <p>Focus question:What does it mean for two events to be equally likely?</p> <p>Connect with last lesson. Was it equally likely for coins to match or not match?</p> <p>Give students quiet time to read Calvin’s story about the penny on the train track.</p> <p>Why is Calvin’s mother suspicious of the coin? What does it mean for a coin to be “fair”? Read the story about names in a hat. Why is each card equally likely to be chosen but not each name? How many more of each name to add to make it equally likely? Have students address Problem 1.4 on page 16- discuss in small groups.</p> <p>Homework #19 on page 20.</p> <p>Close: Pair exercise- # 31 on page 24 of CMP- students can work in pairs or triples to complete.</p>
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Table 22 (continued)

Assessing and advancing questions	What are the possible outcomes? How did doing the experiment help you to visualize the outcomes? What does it mean to model mathematics? In what ways has modeling helped you? We have talked about visualizing many times, is there a connection?
Ways that authority will be shared with students	Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. They have authority to construct their own record keeping system.
Ways that students will be held accountable to each other and the teacher	Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding. Exit slip- What was the mathematical message of this lesson?
Student resources that will be made available	Peers. Pennies. Cups. Question examples hanging on the wall.
Problem-tizing: Ways that students will be challenged in ways that engender genuine uncertainty	Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution. My intent is that through questioning I can help students to make connections between probability and proportional reasoning.

3-5-14- Day 4 I felt like I had to be the authority and begin the class with clarification regarding the percentage of matches with two coins. I don't feel really comfortable about doing that, but I wanted to be sure they all knew there were four outcomes, not three. From there, students worked in groups to define "experimental" and "theoretical" probability. I've been using the words and wanted them to stop and consider their meaning and the relationship between them. I used a Venn diagram to model the two and scribed kids' words in each side. It was clear that they didn't understand theoretical probability nor the relationship between the two words. Using a simulation program for rolling a dice, students watched the experimental probability move toward the theoretical value as the number of rolls increased (sample size). Students struggled to generalize in words what they noticed as the sample size increased due to limited familiarity with vocabulary, I think. I'm hoping as they continue to use the words, that vocab will flow. Finally, I administered a formative assessment completed in pairs. I would classify this quiz as an assessment for learning, as I think the questions helped students make connections (based on their comments during the quiz). A number of students were on a Science trip today, but I felt like the class was eager to engage and willing to struggle through some tough concepts.

The formative assessment that I administered had several functions. First, it acted as a pretest to determine if kids understood that the sum of the probabilities of all events must equal 1. The first question addressed this issue as well as #2, to a lesser degree. Students definitely understood that as evidenced by 100% of students answering that question correctly. Second, I wondered if kids noticed that probability had to be less than or equal to one. #2 addressed this concept and only one pair missed this question. Some students were not thorough in their analysis of problems 3a and b. Several students didn't notice that both scenarios were impossible; a probability of 0.

APPENDIX N

LESSON PLAN AND REFLECTION

(DAY 5)

Table 23. Lesson Plan and Reflection (Day 5)

Procedure	<p>Probability- Day 5 and 6</p> <p>Review homework on day 5 : Students will work in groups for a short time to compare answers. I will circulate and ask questions especially focused on c,d that points to proportional reasoning(3 pts for completion) . Whole group discussion regarding homework grading policy. Plan to increase pts to 10 – all effort, not correctness.</p> <p>Assign pairs of students to solve the Quasar marble task. Remind students to use private think time, then solve using two ways. Probe student thinking as they solve. I will use the tool from the 5 Practices book to order student work on day 6. I will order from simplest to most complex- Students will explain their own work, while the class asks clarifying questions. Drawing connections among representations will be the focus of my questionsl.</p> <p>If time permits,I will explore what they know about compound events to determine the best place to start on Monday. I wonder what record keeping system they are familiar with regarding compound events? I will ask them to find the theoretical probability of tossing 3 heads if they are tossing 3 two-sided coins.</p>
Assessing and advancing questions	<p>What other tasks have you done that relate to this one? Why would you examine the number of blue /red marbles? What conclusion can you draw from this information? Is there another way to represent the information? Could you use this ratio to produce a graph? In what way does unit rate relate to slope?</p>

Table 23 (continued)

Ways that authority will be shared with students	Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. They have authority to construct their own record keeping system.
Ways that students will be held accountable to each other and the teacher	Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.
Student resources that will be made available	Red/blue chips, graph paper, peers, .
Problemsetting: Ways that students will be challenged in ways that engender genuine uncertainty	<p>Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution.</p> <p>My intent is that through questioning I can help students to make connections between probability and proportional reasoning.</p>

The wrap up for the marbles task didn't go very well. The class had two tests in other subjects today and they arrived in a rather restless state. Behavior wasn't the best for some, and for others they didn't contribute much to discussion. Student work was of varied representations which I was pleased with, but the conversation to bring it all together and connect them was only ok. I have done this lesson before and had a much more robust discussion. I'm not sure that this task created as much uncertainty as I would like. It was too easy for them to quickly arrive at a percent. The only thing that made it interesting was asking for two representations. Dennis Mason is not engaging in my class or any class. It is frustrating because he is smart, but none of his teachers has connected with him despite our effort. Students certainly had the opportunity to take authority; explaining their own work and ideas. I'm tired today, perhaps my attitude was apparent to students today and affected their performance.

APPENDIX O

LESSON PLAN AND RELECTION

(DAY 7)

Table 24. Lesson Plan and Reflection (Day 7)

Procedure	<p style="text-align: center;">Probability- Day 7</p> <p>Mini quiz to start class- 1)How do you find the experimental probability that a particular result will occur?2) In an experiment, are 30 trials as good as 500 trials to predict the chances of a result? Explain. 3) Quasar spinner problem.</p> <p>Focus question- How does understanding probability help you to determine a winning strategy?</p> <p>Following the quiz, students will finish their work regarding the probability of flipping three heads, if you toss 3 coins. The groups were very wild on Friday. Make this an individual activity for 5 minutes, then a group activity. Do students know how to use a tree diagram? An organized list? Let students demo both strategies. Discuss in whole group. In class #12, 13 page 40 and 14-17, page 41.</p> <p>Return to the focus question. Discuss in whole group.</p>
Assessing and advancing questions	What other tasks have you done that relate to this one? What strategies did you use for keeping track of your thinking?
Ways that authority will be shared with students	Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. They have authority to construct their own record keeping system.
Ways that students will be held accountable to each other and the teacher	Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.

Table 24 (continued)

Student resources that will be made available	Red/blue chips, graph paper, peers, .
Problemizing: Ways that students will be challenged in ways that engender genuine uncertainty	<p>Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution.</p> <p>My intent is that through questioning I can help students to make connections between probability and proportional reasoning.</p>

Students were unfamiliar with the use of tree diagrams for keeping track of outcomes. I reminded them that they had learned it in fourth grade, but it seemed like a distant memory. They used it to find combinations of food and a spinner/ cube compound event. Students were very eager and stayed on task. They are very competent at explanations for the most part and few avoid the task. No one seems to be hurt by redirection either, a change from the start of the year.

APPENDIX P

LESSON PLAN AND REFLECTION

(DAY 8)

Table 25. Lesson Plan and Reflection (Day 8)

Procedure	<p>Probability- Day 8</p> <p>Mini quiz to start class- NAEP question- labeled question 2. Majority of the class will be centered on the sticky gum problem in small groups. I expect students will solve via an organized list, pictures, and using tree diagrams. Large whole group discussion will encourage students to critique the reasoning of others and justify their reasoning. If time permits, I will encourage students to make a Table 4.2nd generalize a solution.</p>
Assessing and advancing questions	What other tasks have you done that relate to this one? What strategies did you use for keeping track of your thinking? Can you generalize a solution for any number of children with two colors of gum balls?
Ways that authority will be shared with students	Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. They have authority to construct their own record keeping system.
Ways that students will be held accountable to each other and the teacher	Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.
Student resources that will be made available	Red/blue chips, graph paper, peers, .

Table 25 (continued)

<p>Problem etizing: Ways that students will be challenged in ways that engender genuine uncertainty</p>	<p>Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution.</p> <p>My intent is that through questioning I can help students to make connections between probability and algebraic reasoning.</p>
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Day 8- Tuesday

The first task was a quiz from the released NAEP items. Students must take a position and provide an explanation. Then small groups addressed the sticky gum problem. Student groups used organized lists or reasoning based on “worst case” scenarios (alternating colors) to determine the number of pennies that Mrs. Jones would need . One student did an amazing job at the end of the class in generalizing the pattern; noticing that the number of pennies would always be odd for 2 colors of gum balls. I’m quite sure that most of the class didn’t follow his thinking but that’s ok with me. I feel as though the students that are most capable of thinking abstractly had the opportunity and encouragement to do it and those that are developing that capacity had someone to model it for them. I’m sure they noticed that I valued that capacity. Many students were unsure of the way to enter the problem and struggled to get going so I’m quite sure that problematizing was achieved. Students authored their own solutions using whatever model they chose. I’m thinking about where to go from here. I have three more days of instruction and want to be sure they could solve problems related to compound events. I’m a little concerned that no groups in this class used a tree diagram as a model.

I’m feeling more than a little stressed. This lesson with the modified sticky gum problem bridges the way to learning about two-stage probability with and without replacement. Based on upcoming standardized testing, I will need to stop this unit and review test directions for at least 3 days prior to the test. I also haven’t even touched on area models at all. I don’t want to tell students that the procedure for multiplying the probabilities of each event so I’ll follow CMPs method of using an area model so that kids can make sense of it. The model also lends itself to additional experience with proportional reasoning. It will take more time, but it will be worth it if kids remember it through sense making.

APPENDIX Q

LESSON PLAN AND REFLECTION

(DAY 9)

Table 26. Lesson Plan and Reflection (Day 9)

Procedure	<p>Probability- Day 9</p> <p>*The sticky gum problem introduced the idea of more than one event happening. We found the number of pennies that Mrs. Jones would need to spend, but not the probabilities or frequencies of each event. Today we will explore the ways that an area model can help us make sense of a situation so that we can analyze probabilities.</p> <p>* Note that students have used two strategies: o lists and tree diagrams. Today we will add a third strategy. (area models)</p> <p>* Start 4.1 pg. 72- Have students read silently, then address in small groups. Go through section focusing on the relationships between area and proportional reasoning.</p> <p>* After discussion, let students begin homework (#1-4 pg 80 due Thurs), (#5 pg. 81 due Friday)</p>
Assessing and advancing questions	<p>What does each partition in the side of the square represent? What does the area of each section represent? How many regions are there? Why? What do you notice about the sections when there is an equal number of each color? When it is different? What other numbers of each color might this same table represent?</p>
Ways that authority will be shared with students	<p>Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. I am removing some authority today by insisting on one representation, but it is necessary.</p>

Table 26 (continued)

Ways that students will be held accountable to each other and the teacher	Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.
Student resources that will be made available	Red/blue chips, graph paper, peers, .
Problem etizing: Ways that students will be challenged in ways that engender genuine uncertainty	<p>Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution.</p> <p>My intent is that through questioning I can help students to make connections between probability and algebraic reasoning.</p>

Day 9- Thursday- CMP 4.1 Students were very lively today. I suspect that Bob understands more than he seems. He loves any way to get attention. I'll check tomorrow. I like the way that CMP uses and area model to teach compound events. ..better than just telling students to multiply as the HM text does. I wonder if tomorrow, students will notice that they are multiplying the probability of one bucket with the other. Problematizing was really noticeable today. Many students actually said that they didn't know where to start. Providing the idea that a 10 x 10 grid might be used to help them was my way of helping them. I wonder if they will come into class tomorrow , with a connection to proportional reasoning. Nya didn't see the value of the area model today...wonder if she will tomorrow. Nikki is still not taking risks. I think it is time for a formative assessment to determine if all are understanding.

APPENDIX R

LESSON PLAN AND REFLECTION

(DAY 10)

Table 27. Lesson Plan and Reflection (Day 10)

<p>Procedure</p> <p>Resources; need blanks of spinners, 10 sided number cubes, blocks in a bag.</p>	<p>Probability- Day 10</p> <p>*Review homework(#1-4 pg 80 due Thurs) What solutions/problems did individuals have? Compare solutions within groups first then ask questions.</p> <p>* Introduce Section 4.3 One-and-One Free Throws</p> <p>Focus question: how is an area model for the free throw situation unlike ones we've used before?</p> <p>Begin by reading the scenario so that students understand what it means to have a one and one free throw. Follow questioning on pg. 76 to be sure they understand the context. Ask students to address 4.3 part B. We will use only an area model. However, students will design simulations. Each model must address a miss on the first attempt, a hit followed by a miss and a hit followed by a hit. Does yours?</p> <p>How would you simulate this with a ten-sided number cube? A spinner? Blocks in a bag? Assign student groups the model they will use.</p>
<p>Assessing and advancing questions</p>	<p>What does each partition in the side of the square represent? What does the area of each section represent? How many regions are there? Why? What do you notice about the sections when there is an equal number of each color? When it is different? What other numbers of each color might this same table represent?</p>
<p>Ways that authority will be shared with students</p>	<p>Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. I am removing some authority today by insisting on one representation, but it is necessary.</p>

Table 27 (continued)

Ways that students will be held accountable to each other and the teacher	Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.
Student resources that will be made available	Red/blue chips, graph paper, peers, .
Problem etizing: Ways that students will be challenged in ways that engender genuine uncertainty	<p>Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution.</p> <p>My intent is that through questioning I can help students to make connections between probability and algebraic reasoning.</p>

Day 10. Thursday Dennis Mason's group was filmed today. He said he didn't understand, but seems to make no effort to ask questions. Part way through the class, I asked him to come to the doc projector and explain his thinking. I gave him extra time and told his team they were accountable for helping him to understand, but still he didn't make any effort. I asked Ed to come up with him to support him and Ed did all the talking. I was quite frustrated with his behavior. I have tried to reach him, but seem not to make any progress at all. Most often he completes not homework and participates very little in class. When he does complete work, he seems to have reasonable number sense.

Overall the class was confused by the area model 4.2 problem wherein they needed to work backwards. The problem was excellent because it brought to light their lack of connection to percentages and proportional reasoning (with probability). I will plan to do some direct instruction tomorrow and determine specific areas of partial understanding. Also I'll give a formative assessment to assess progress.

APPENDIX S

LESSON PLAN AND REFLECTION

(DAY 11)

Table 28. Lesson Plan and Reflection (Day 11)

<p>Procedure</p> <p>Resources; need blanks of spinners, 10 sided number cubes, blocks in a bag.</p>	<p>Probability- Day 11</p> <p>Focus question: how is an area model for the free throw situation unlike ones we've used before?</p> <p>Check to be sure students understand the relationship between an array and area model. Help them make the connection between finding area, and the area model.</p> <p>Yellow face cube task</p>
<p>Assessing and advancing questions</p>	<p>What does each partition in the side of the square represent? What does the area of each section represent? How many regions are there? Why? What do you notice about the sections when there is an equal number of each color? When it is different? What other numbers of each color might this same table represent?</p> <p>How does proportional reasoning help you complete the yellow face cube task?</p>
<p>Ways that authority will be shared with students</p>	<p>Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. I am removing some authority today by insisting on one representation, but it is necessary.</p>

Table 28 (continued)

Ways that students will be held accountable to each other and the teacher	Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.
Student resources that will be made available	Draw cube on white board, graph paper, peers, Drawing area model on white board
Problem etizing: Ways that students will be challenged in ways that engender genuine uncertainty	<p>The yellow faced cube problem will assess whether students are making a connection to proportional reasoning.</p> <p>My intent is that through questioning I can help students to make connections between probability and algebraic reasoning.</p>

Day 11- Friday- Last day of probability before PSSA tests. I fretted about the structure of this class last night, recognizing that there was a lot of confusion at the end of the class yesterday. I know I need to help them connect the area model to their prior knowledge of area. I don't see a good way to do it without me doing most of the telling. I want to be the authority today, so that I leave no question about my thinking of a way to use an area model for the purpose of modeling probability of two independent events. Decided to lay out my thinking in stages going from left to right across the board. The leftmost picture, the starting point will be a square noting that $A = l \times w$. Next, I'll show a small square and note the area is $3 \times 2 = 6$ in sq. Next, I'll show one bucket's marbles represented along the length, demanding that the rectangle be separated into fifths. I'll use a clear sheet to overlay the marbles of other bucket on top ($1/4$ of the bucket for each color). Now to find Blue/blue we have $1/5$ of $1/4$ or $1/20^{\text{th}}$. Finally, I'll give kids a spinner/marble bag problem to do independently

This lesson seemed to work well. All but 2 kids solved the problem successfully alone. Dennis refused to write anything, even when I prompted him and probed a little. He clearly is being passive aggressive. I collected their homework then gave the NAEP question (how many yellow sides?) as a quiz. It is a good place to take a break for the PSSAs.

APPENDIX T

LESSON PLAN AND RELECTION

(DAY 12)

Table 29. Lesson Plan and Reflection (Day 12)

<p>Procedure</p> <p>Resources; area model that we constructed for Making Purple, last week.</p>	<p style="text-align: center;">Probability- Day 12</p> <p>Focus question: How can you use experimental or theoretical probabilities of a compound event to predict the number of times one particular combination will occur out of any given number of repetitions of the event?</p> <p>Review how to analyze a two-stage outcome using an area model. Have students turn to pg. 75 in CMP. Review our discussion of last week. Are purple and not purple equally likely? How might you figure it out exactly? For Spinner A, what is the likelihood of getting red? How is this represented on the square? Distribute the area model that we agreed to for that scenario.</p> <p>Address question D. Let students work in pairs to complete. Every person must turn in a written response. Following student completion, engage students in a discussion of their thinking. If time permits, begin the next lesson: regarding one and one free throws.</p> <p>Homework #8, page 82. Complete written explanation is expected.</p>
<p>Assessing and advancing questions</p>	<p>What does each partition in the side of the square represent? What does the area of each section represent? How many regions are there? Why? What do you notice about the sections when there is an equal number of each color? When it is different? What other numbers of each color might this same table represent?</p>
<p>Ways that authority will be shared with students</p>	<p>Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. I am removing some authority today by insisting on one representation, but it is necessary.</p>

Table 29 (continued)

Ways that students will be held accountable to each other and the teacher	Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.
Student resources that will be made available	Red/blue chips, graph paper, peers, .
Problemizing: Ways that students will be challenged in ways that engender genuine uncertainty	<p>Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution.</p> <p>My intent is that through questioning I can help students to make connections between probability and algebraic reasoning.</p>

Day 12- This was our first day after the pSSA break, so I resumed instruction of the Making Purple game and followed the lesson plan exactly. Two groups were working on developing a relationship between profit and number of people at the end of the class. I will begin with that tomorrow. The students were quiet while Sydney explained her rationale because most got the same answer. Bob didn't but then seemed to understand once Mrs C explained. There was lots of good small group discussion that I missed because my mic wasn't on. ☹ Not sure if we got any sound at all.

APPENDIX U

LESSON PLAN AND REFLECTION

(DAY 13 /14)

Table 30. Lesson Plan and Reflection (Day13/14)

<p>Procedure</p> <p>Resources; need blanks of spinners, 10 sided number cubes, markers</p> <p>Handout with questions to answer.</p>	<p>Probability- Day 13 and 14</p>
	<p>*Focus question: How is an area model for the one and one free-throw situation like or unlike the area model for the Making Purple game?</p> <p>(see CMP pg. 204) Begin by describing what it means to have a one and one free throw situation. Have a student demonstrate with the classroom hoop. Pose to the class: Which score is most likely to happen? How might we figure it out? How could we SIMULATE a situation to generate experimental data about the likelihood of each result using spinners, cubes or blocks? What does SIMULATE mean? The discussion should raise the question of how a simulation will handle the fact that getting the second shot depends on whether the first free throw is made. See page 204 for examples of student strategies. Have students demo their suggestions. Offer students the blank spinners/markers. Make sure they understand to record the score of each trial(0,1,2)</p> <p>Combine all the experimental data and find an overall class experimental probability.</p> <p>Provide an area model that represents the one-and –one free throw scenario. Ask students to explain in writing,1) how the area model helps to find the theoretical probability and 2) How it is different than the Making Purple area model. What is the theoretical probability of scoring 0, 1, 2?</p> <p>Wrap up- Engage is a discussion; How does the experimental probability compare to the theoretical probability?</p> <p>Homework pg. 84,85 #14 and 15.</p>

Table 30 (continued)

Assessing and advancing questions	What does each partition in the side of the square represent? What does the area of each section represent? How many regions are there? Why? What do you notice about the sections when there is an equal number of each color? When it is different? What other numbers of each color might this same table represent? How does the model represent a “miss”.
Ways that authority will be shared with students	Students must take charge and figure out what they don’t know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. I am removing some authority today by insisting on one representation, but it is necessary.
Ways that students will be held accountable to each other and the teacher	Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.
Student resources that will be made available	Grid paper, graph paper, peers, .
Problem-tizing: Ways that students will be challenged in ways that engender genuine uncertainty	Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution. My intent is that through questioning I can help students to make connections between probability and proportional reasoning.

Day 13- (mon) The task was to label and make sense of a partially completed area model that represented one and one free throws. They made some sense of the model but it was clear that they don’t see the idea of taking a fraction or percent of a fraction. ...one probability times another probability. A few noticed that the likelihood of getting 2 pts was 36% but no one said anything about multiplying .60 by .60 to get it. The break in continuity of thinking seemed to stem from a partial understanding of the two events they were modeling. Throw 1 and Throw 2 were not clearly marked on the model with distinctions for what the outcomes might be for each. Bob is so anxious for attention he wants to talk whether he has something justifiable or not. Tomorrow students will simulate the context using spinners and 10 sided cubes. I will come back to the area model at the conclusion of the simulation to compare exp and theor probability.

APPENDIX V

LESSON PLAN AND REFLECTION

(DAY 15)

Table 31. Lesson Plan and Reflection (Day 15)

<p>Procedure</p> <p>Resources; need blanks of spinners, 10 sided number cubes, Labsheet 4.3</p>	<p>Probability- Day 15</p> <p>Begin with a review of homework, problems 14,15 on page 85 (the cave problem that asks to match an area model and the picture of the paths to the caves). Have students vote on their choice of path pictures. Then I will choose a student to justify his reasoning- other students will be encouraged to challenge and reason. I will direct the discussion as necessary to help students make meaning of the size of the pieces of the area model and the meaning of the labels. I will then have students complete the exit slip re: area models and simulations. Finally students will make generalizations about the exp prob and theor prob of the one-an-one free throw simulation they did yesterday. Short discussion of some of the things they observe (exp data is much like theoretical data,</p>
<p>Assessing and advancing questions</p>	<p>What does each partition in the side of the square represent? What does the area of each section represent? How many regions are there? Why? What do you notice about the sections when there two paths at a decision point? When it is different? What maps can you rule out based on the area model for choice one?</p>
<p>Ways that authority will be shared with students</p>	<p>Students must take charge and figure out what they don't know and seek understanding. Students will do the majority of talking and question asking of peers. Students will be pressed for evidence. I am removing some authority today by insisting on one representation, but it is necessary.</p>
<p>Ways that students will be held accountable to each other and the teacher</p>	<p>Team members will be accountable to respond to peers and me. I will use AT to encourage discussion and check for understanding. Written work will be evidence of student effort and understanding.</p>

Table 31 (continued)

Student resources that will be made available	Grid paper, graph paper, peers, .
Problem etizing: Ways that students will be challenged in ways that engender genuine uncertainty	<p>Selected problems are new and require making connections to prior learning and understanding. I seek to establish uncertainty and a necessity to persevere to work toward solution.</p> <p>My intent is that through questioning I can help students to make connections between probability and proportional reasoning.</p>

APPENDIX W

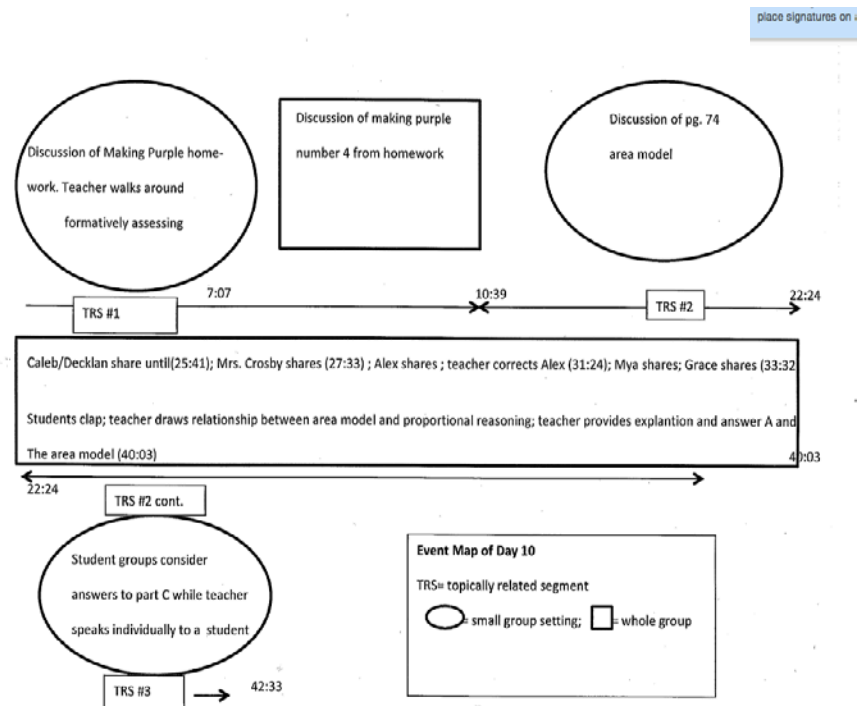


Figure 21. Event Map

APPENDIX X

STUDENT WORK- MARBLES TASK

Bag Y gives you the best chance of picking a blue marble because of all the bags there is a greater number of red marbles than bag Y.

Bag X <u>55</u>	Bag Y <u>20</u>	Bag Z <u>75</u>
--------------------	--------------------	--------------------

Bag X has 55 more red marbles than blue. Bag Y has only 20 more red marbles. Bag Z has 75 more red marbles. Therefore, Bag Y has the best chance. ✓

75
 -25

 50

40
 -20

 20

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Figure 22. Student 1 Solution

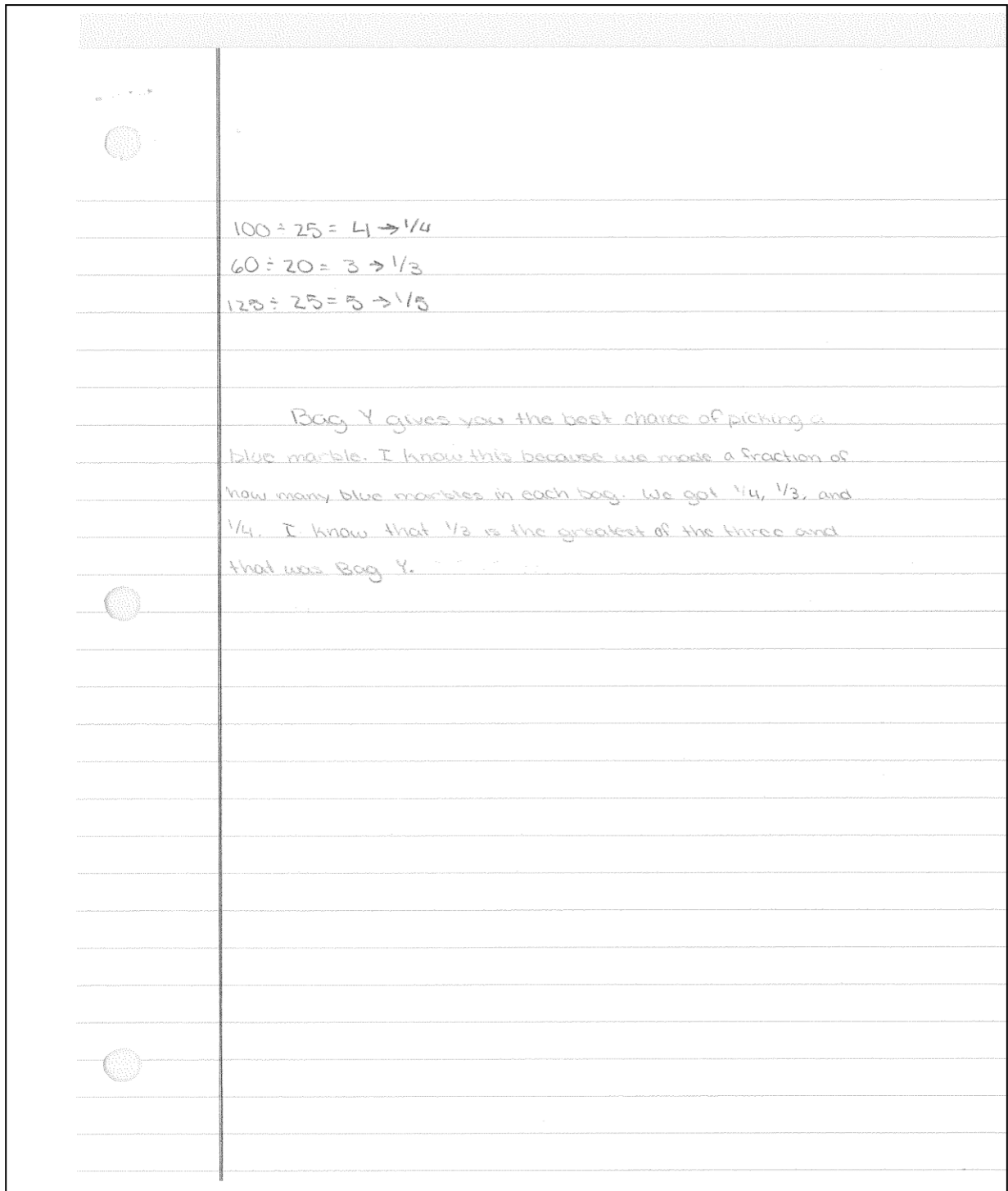


Figure 23. Student 2 Solution

I would choose this bag because there is a 33% chance that a student would pick a blue marble. I know this because the total number of marbles in bag Y is 60. Then I divided 100 by 60 to see what I would need to multiply 20 and 40 by to get 100%. $100 \div 60 = 1.\overline{66}$. I then multiplied $20 \times 1.\overline{66} = 33.\overline{2}$. The percentage of blue marbles is 33%. Then I multiplied $40 \times 1.\overline{66}$ and got 66.4. The number of red marbles is 66%.

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Figure 24. Student 3 Solution

$\begin{array}{r} 75 \text{ red} \\ 25 \text{ blue} \\ \hline 100 \end{array}$ $\frac{25}{100}$ 25%	$\begin{array}{r} 40 \text{ red} \\ 20 \text{ blue} \\ \hline 60 \end{array}$ $100 \div 60 = 1.\overline{66}$ $60 \times 1.\overline{66} = 99.6 \rightarrow 100$ $20 \times 1.\overline{66} = 33.2$ $40 \times 1.\overline{66} = 66.4$ 33%	$\begin{array}{r} 100 \text{ red} \\ 25 \text{ blue} \\ \hline 125 \end{array}$ $100 \div 125 = .8$ $100 \times .8 = 80\%$ $25 \times .8 = 20\%$
--	---	--

Figure 25. Student 4 Solution

1st:

Bag X

$$\frac{25 \text{ blue}}{100 \text{ Total}} = 25\% \text{ chance}$$

Bag Y

$$\frac{20 \text{ blue}}{60 \text{ total}} = 33\% \text{ chance}$$

Bag Z

$$\frac{25 \text{ blue}}{125 \text{ total}} = 20\% \text{ chance}$$

2nd:

$$\frac{25 \text{ blue}}{75 \text{ red}} \div \frac{5}{5} = \frac{5}{15} \div \frac{5}{5} = \left(\frac{1}{3}\right) \text{ chance}$$

$$\frac{20 \text{ blue}}{40 \text{ red.}} \div \frac{20}{20} = \left(\frac{1}{2}\right) \text{ chance} \text{ Best!}$$

$$\frac{25 \text{ blue}}{100 \text{ red}} \div \frac{5}{5} = \frac{5}{20} \div \frac{5}{5} = \left(\frac{1}{4}\right) \text{ chance}$$

Figure 26. Student 5 Solution

APPENDIX Y

SUMMARY OF QUESTION TYPES ASKED BY INSTRUCTIONAL DAY

Table 32. Summary of Question Types Asked by Instructional Day

Day	Gathering information	Insert Terminology	Explore Mathematical Meaning/ Relationships	Probing	Generate Discussion	Link Apply	Extend Thinking	Orient Focus	Establish Context
1	2	-	3	5	11	1	8	1	-
2	-	1	3	3	10	3	4	2	-
3	7	3	8	6	18	-	-	2	1
4	9	6	8	14	16	3	8	2	-
5	1	-	3	1	7	-	-	-	-
6	5	1	7	11	11	1	1	7	1
7	8	1	6	10	18	-	3	2	1
8	1	-	6	9	8	-	3	-	-
9	8	-	11	8	11	-	2	-	-
10	3	-	9	17	11	-	2	3	-
11	3	-	3	4	1	-	-	2	-
12	-	-	-no audio	-	-	-	-	-	-
13	2	-	6	17	11	-	-	1	-
14	6	4	13	11	4	-	2	-	-
15	-	-	3	2	3	-	-	-	-
Total	55	15	88	118	140	8	33	22	3

Reference: Boaler & Brodie, 2004

APPENDIX Z

EVENT MAPS BY DAY

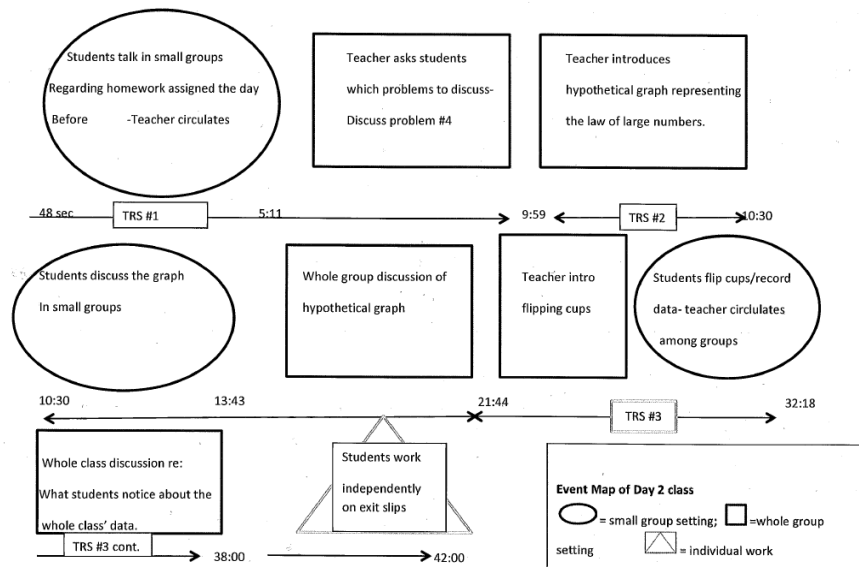


Figure 27. Event Maps by Day 1

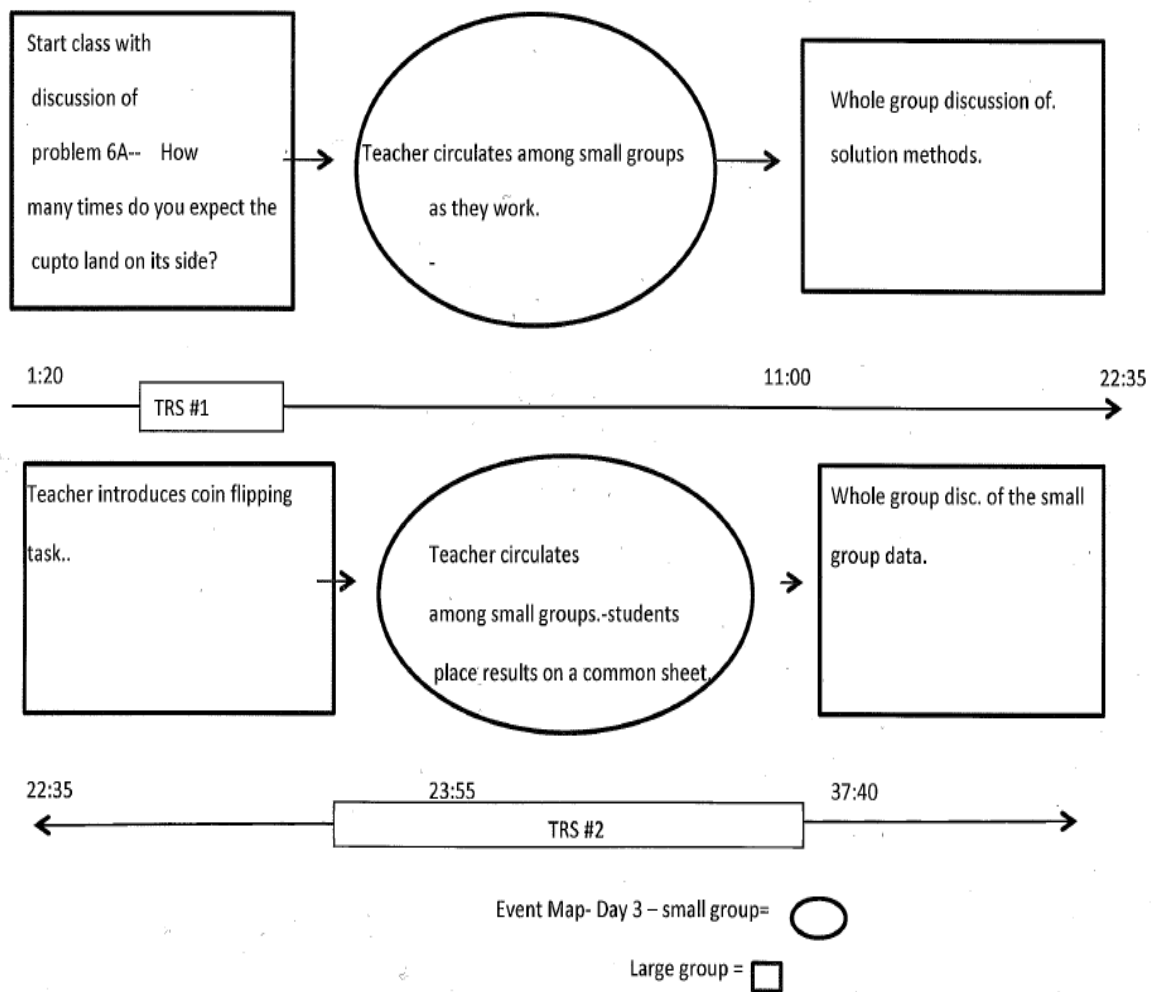


Figure 28. Event Maps by Day 2

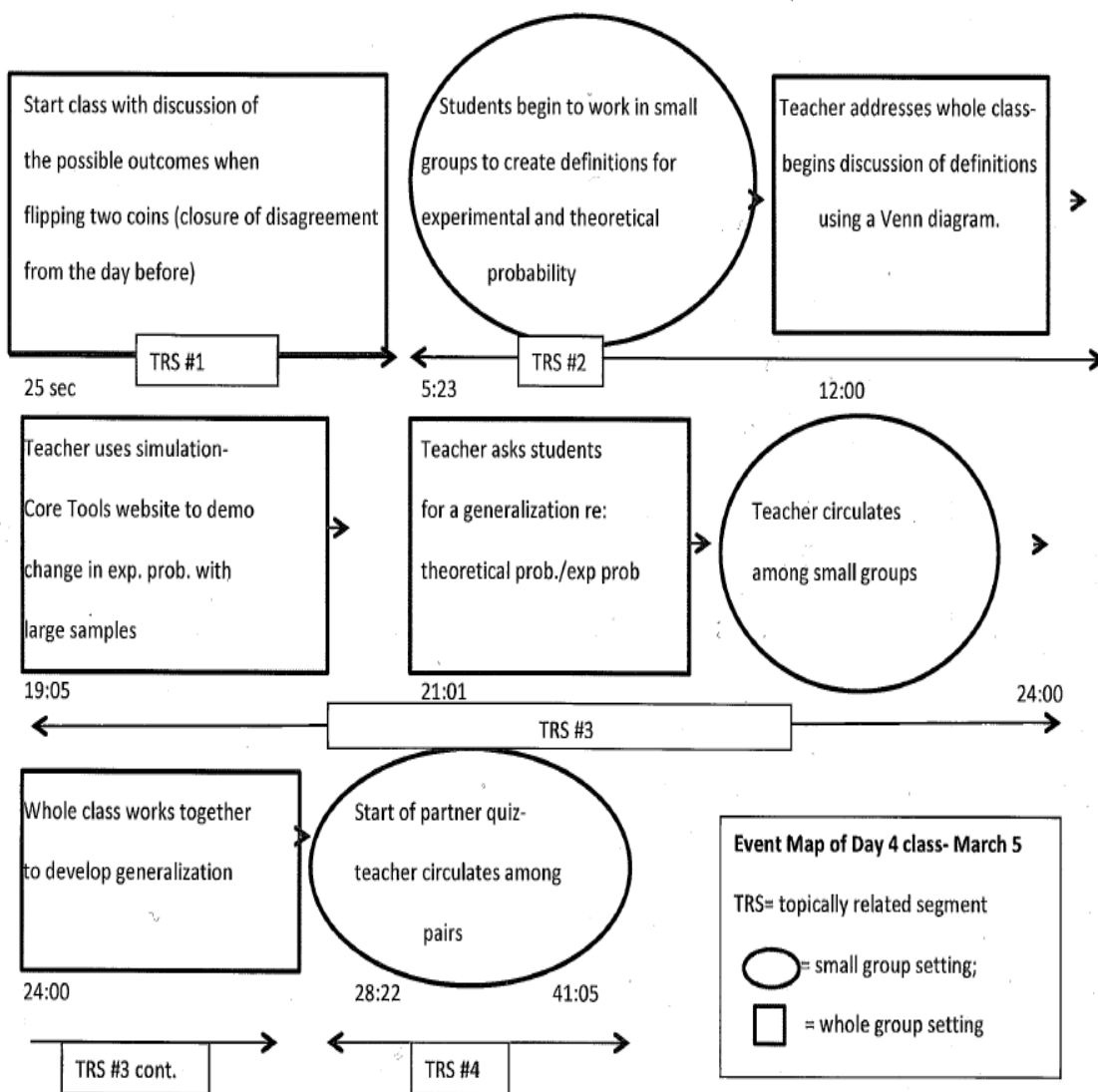


Figure 29. Event Maps by Day 3

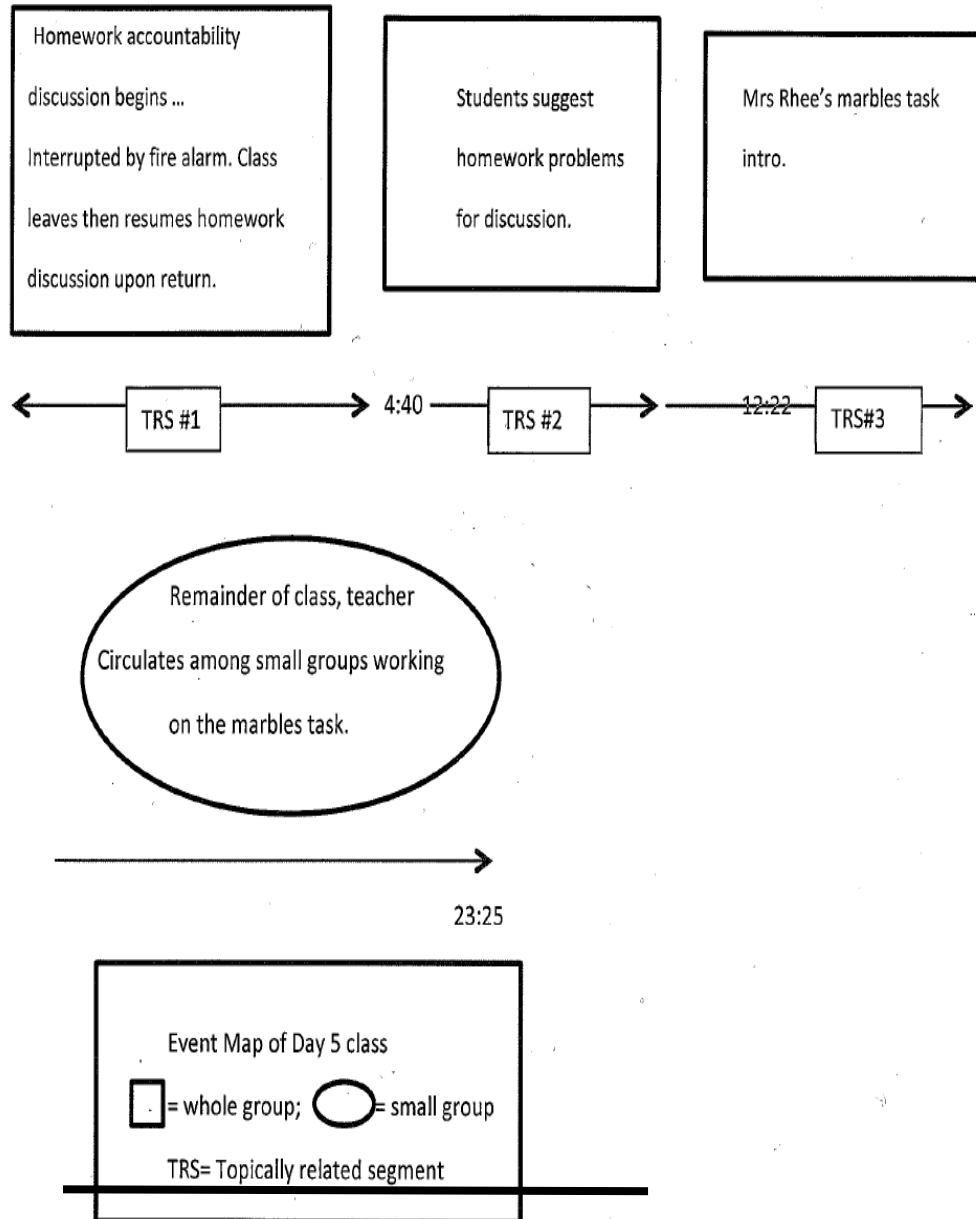


Figure 30. Event Maps by Day 4

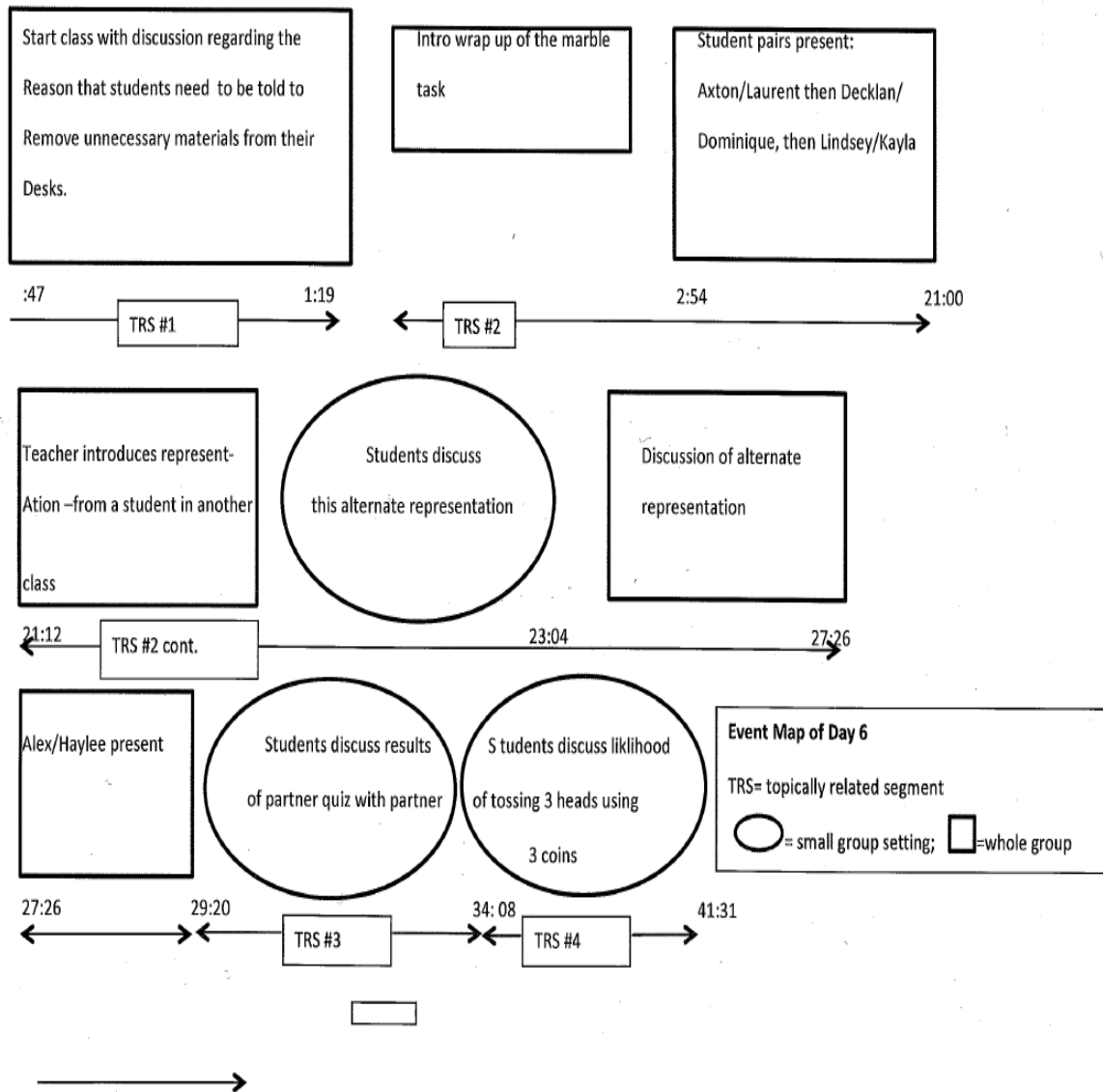


Figure 31. Event Maps by Day 5

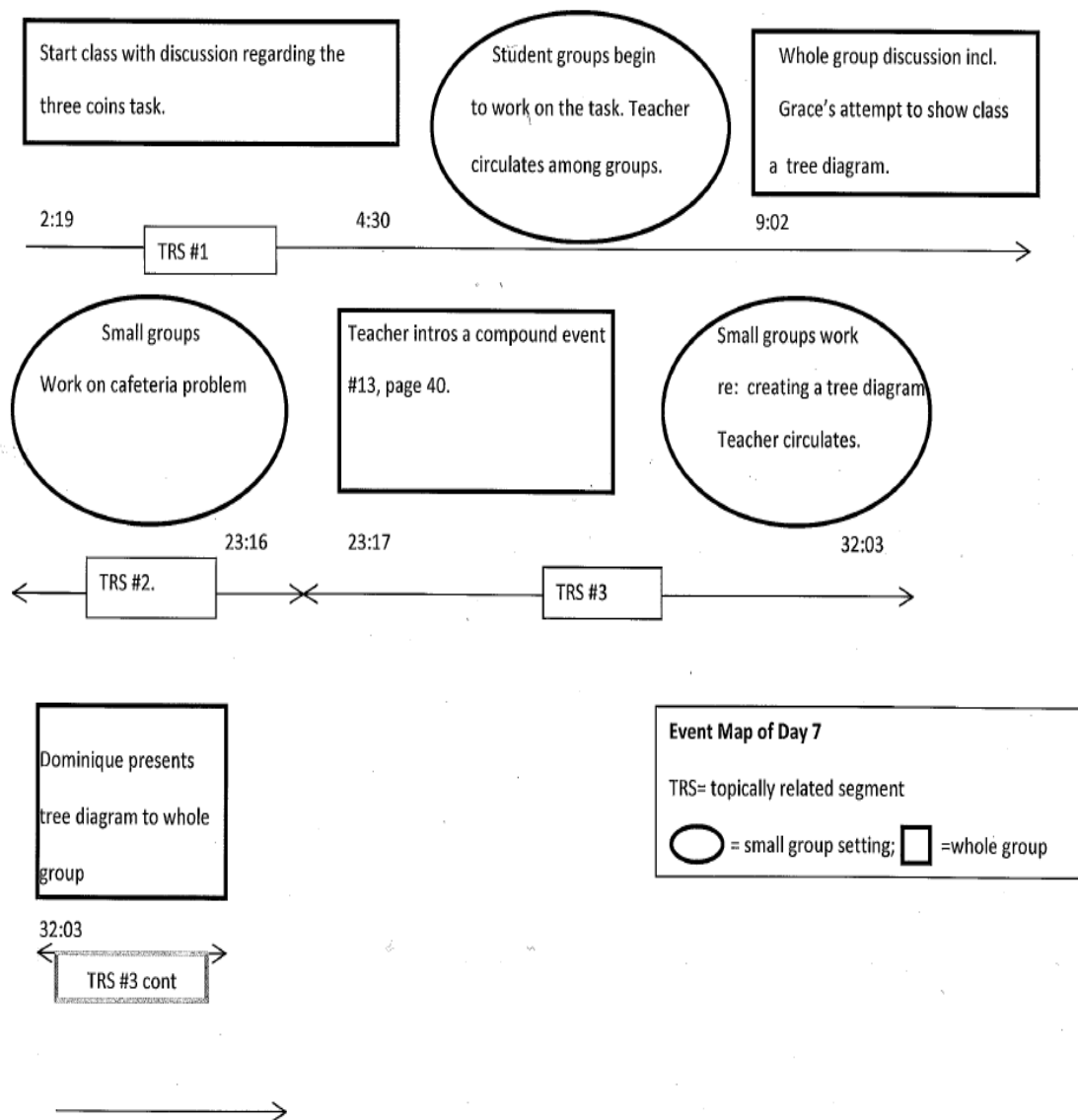


Figure 32. Event Maps by Day 6

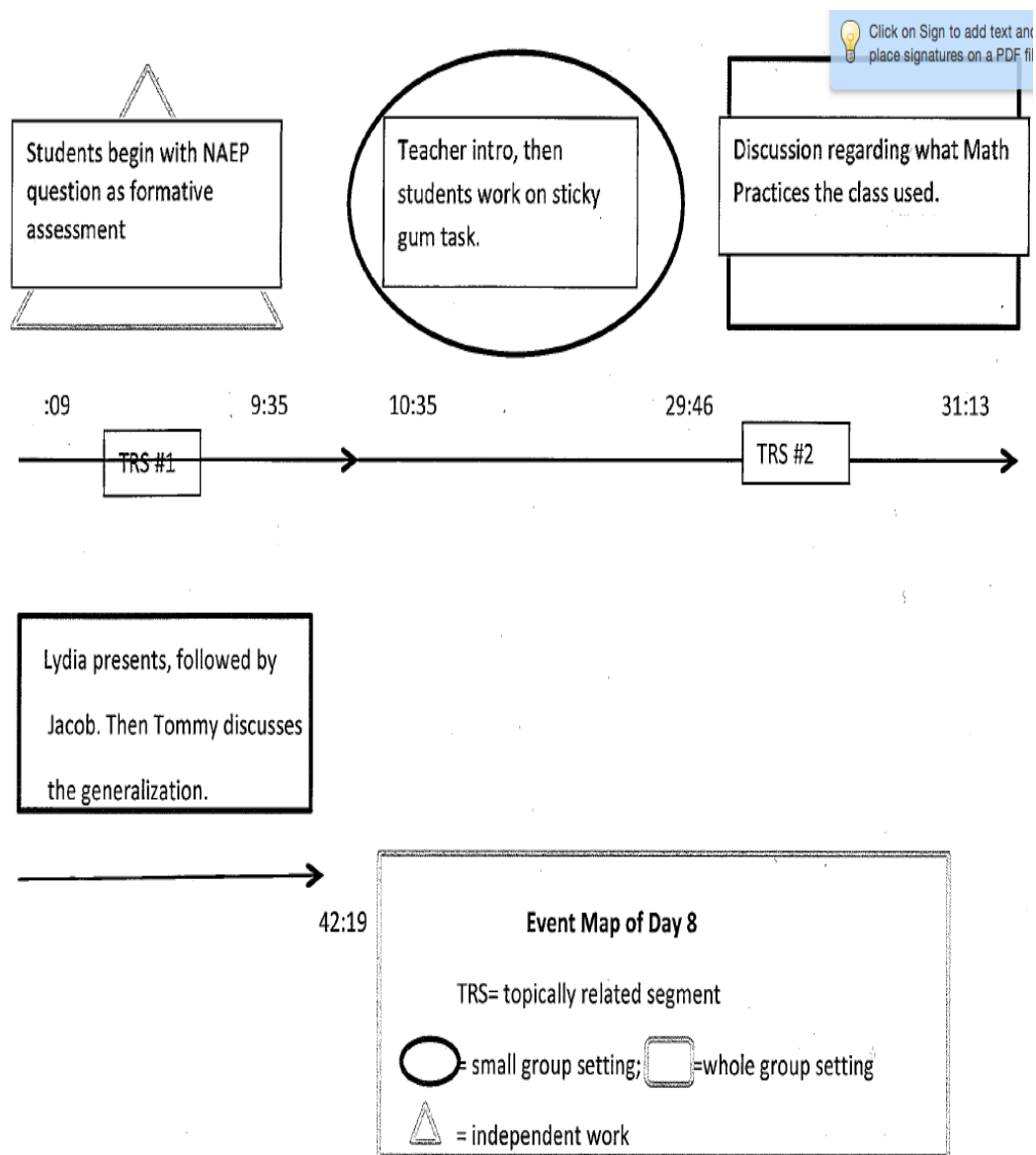


Figure 33. Event Maps by Day 7

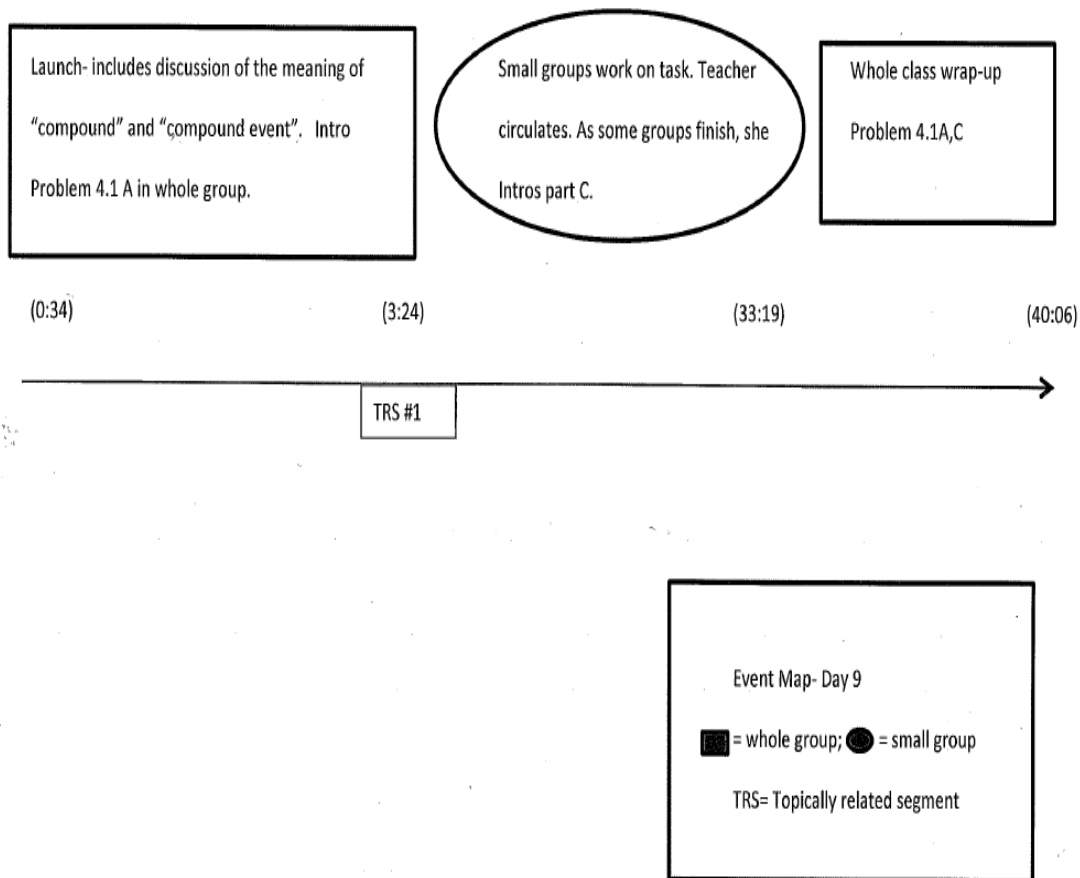


Figure 34. Event Maps by Day 8

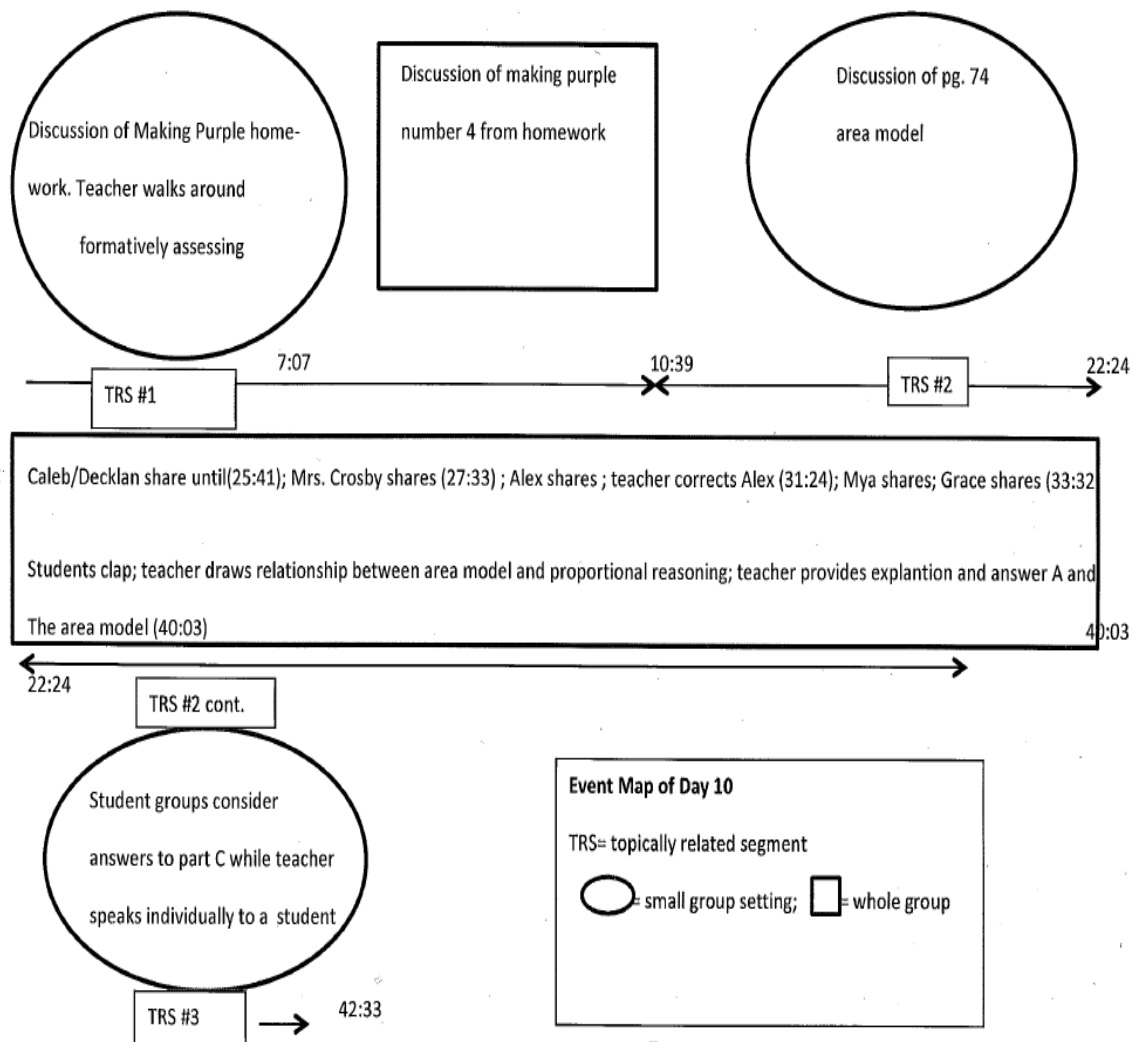


Figure 35. Event Maps by Day 9

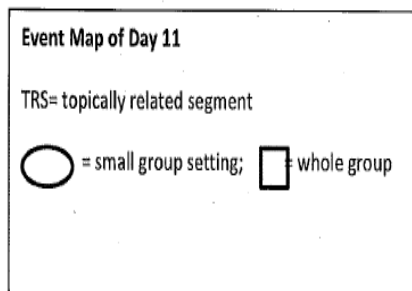
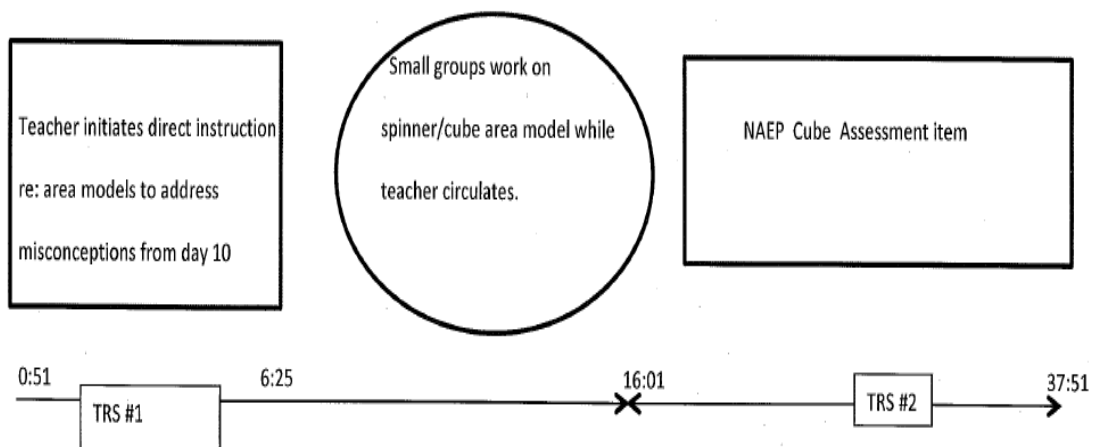


Figure 36. Event Maps by Day 10

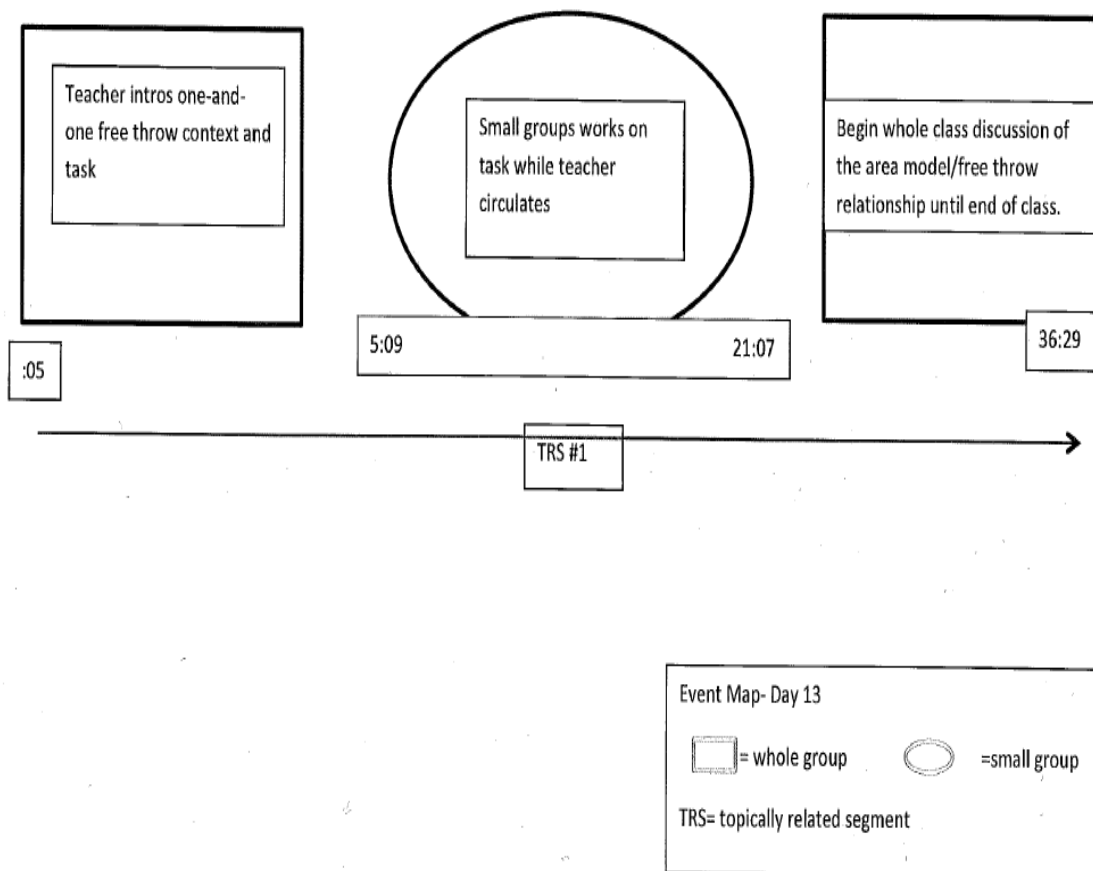


Figure 37. Event Maps by Day 11

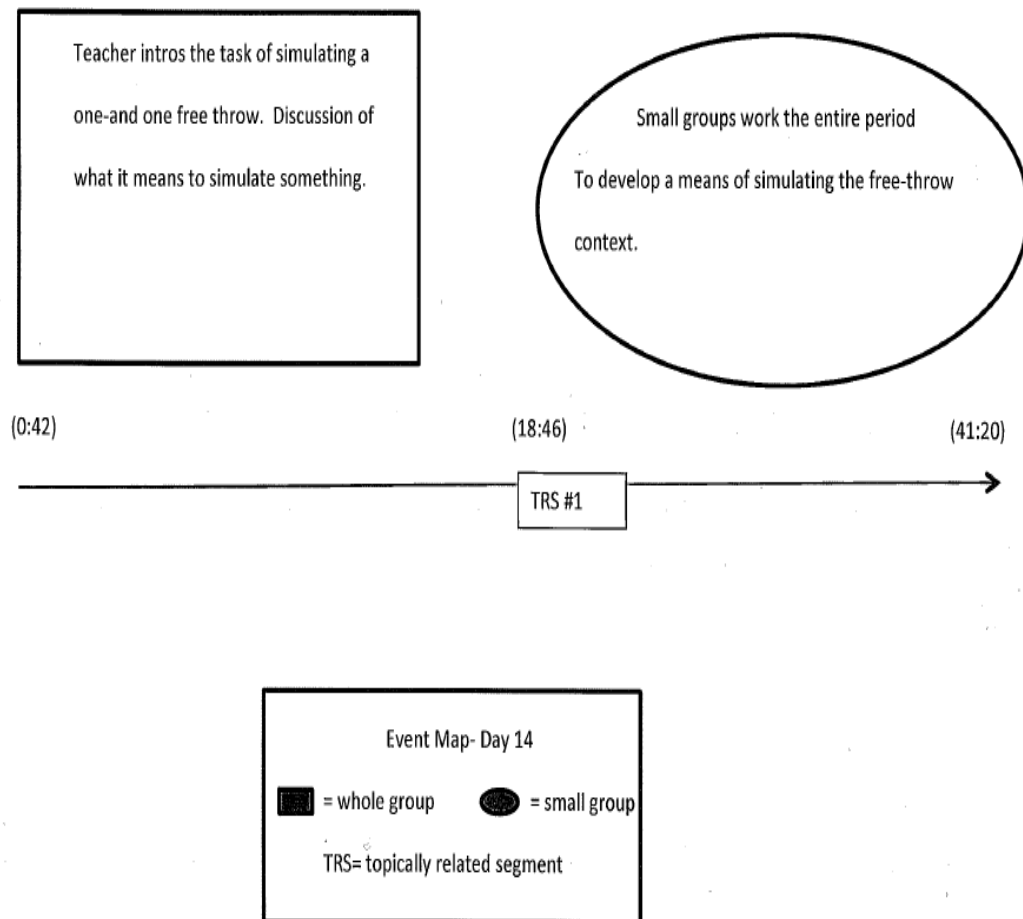
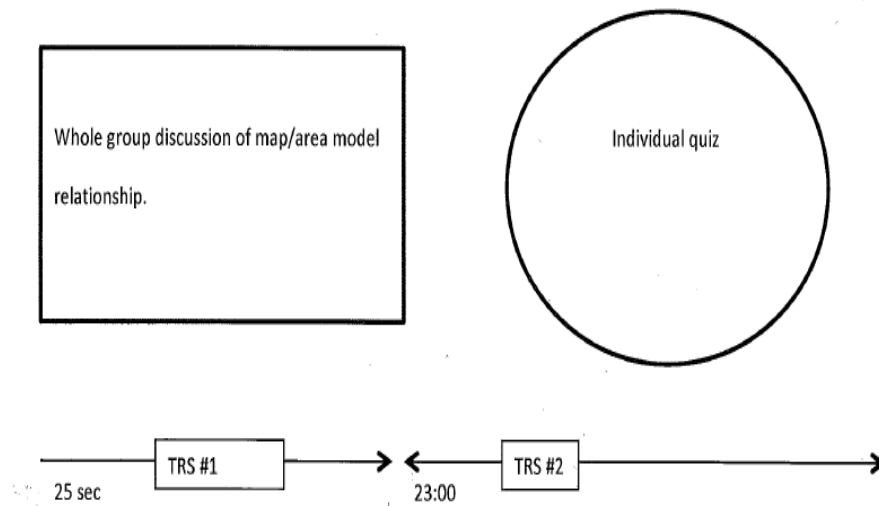


Figure 38. Event Maps by Day 12



Event Map of Day 15 class

TRS= topically related segment

○ = small group setting; □ = whole group setting

Figure 39. Event Maps by Day 13

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